Part I

Quiz 2013: module 5

1.1. Question 21

The correct answer is F. Typically speaking, if we have nodes t_n, t_{n+1}, we won't have access to the nodes in between them.

1.2. Question 23

The correct answer is C. If $f(y_{n+1})$ is used, it's implicit; otherwise, it's not.

1.3. Question 24

The correct answer is G. Remember that the order of the global truncation error is always one lower than the order of the local truncation error. Thus, it'll be of the order $O\left(\Delta t^3\right)$. Note that this means that if Δt is halved, the error will decrease by a factor $(1/2)^3 = 1/8$, and thus the error will now be $\mathcal{E}/8$.

1.4. Question 25

The correct answer is H. We can rewrite

$$y'' - 0.5y' + y = 0$$

as (using $z_1 = y$, $z_2 = y'$ and $z_3 = y''$)

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} z_2 = y' \\ z_3 = y'' \end{bmatrix}$$

leading to

$$\begin{aligned} z_1' &= z_2 \\ z_2' &= 0.5y' - y = 0.5z_2 - z_1 \end{aligned}$$

The initial conditions become

$$\begin{aligned} z_1(0) = y(0) = 0 \\ z_2(0) = y'(0) = 1 \end{aligned}$$

Then, to compute y', we use the fact that $y' = z_1'$, and thus we merely need to compute $z_2(0.1)$. For that, we use the forward step:

$$z_2(0.1) = z_2(0) + \Delta t z_2'(0)$$

with

$$z_2'(0) = 0.5z_2(0) - z_1(0) = 0.5 \cdot 1 - 0 = 0.5$$

so that we get

$$z_2(0.1) = z_2(0) + \Delta t z_2'(0) = 1 + 0.1 \cdot 0.5 = 1.05$$

1.5. Question 26

The correct answer is E. Remember that

$$\eta_{i+1} = \sum_{k=1}^{s} a_{s-k}\eta_{i+1-k} + h\sum_{k=0}^{s} b_{s-k} f_{i+1-k}, \quad i = 0,1,\ldots,n-1$$

combined with

$$\beta^s = \sum_{k=1}^{s} a_{s-k}\beta^{s-k} + hc\sum_{k=0}^{s} b_{s-k}\beta^{s-k}$$

Now, compare

$$y_{n+1} = y_n + \Delta t\left[\alpha f_n + (1-\alpha) f_{n-1}\right]$$

with these formulas. Then we have (from the first equation) $s = 2$ (as for the second sum, we must go down to f_{i-1}, requiring $k = 2$, and thus $s = 2$). Additionally, we have:

- for $k = 0$:
  - that $b_{2-0} = b_2 = 0$ (as we'd get $f_{i+1-0} = f_{i+1}$, which doesn't appear in our formula);
- for $k = 1$:
  - that $a_{2-1} = a_1 = 1$ (as we get $\eta_{i+1-1} = \eta_i$, which appears in our formula);
  - and $b_{2-1} = b_1 = \alpha$ (as we get $f_{i+1-1} = f_i$, which has coefficient α in our formula)
- for $k = 2$:
  - that $a_{2-2} = a_0 = 0$ (as we get $\eta_{i+1-2} = \eta_{i-1}$, which does not appear in our formula);
  - that $b_{2-2} = b_0 = (1-\alpha)$ (as we get $f_{i+1-2} = f_{i-1}$, which has coefficient $(1-\alpha)$ in our formula).

Thus, we get

$$\begin{aligned} \beta^2 &= a_1\beta^1 + a_0\beta^0 + z\left(b_2\beta^2 + b_1\beta^1 + b_0\beta^0\right) = \beta + z\left(0\cdot\beta^2 + \alpha\beta + (1-\alpha)\right) = \beta + \alpha z\beta + (1-\alpha)z \\ 0 &= \beta^2 - (1+\alpha z) - z(1-\alpha) \end{aligned}$$

Alternatively, you can remember that you multiply each derivative with β to the power of the order of the derivative (i.e., ys are simply multiplied with $c^0 = 1$, fs are multiplied with $c^1 = c$, f's are multiplied with c^2, etc.), and that the terms corresponding to 'oldest' step appearing in the scheme is replaced with 1 (in this case f_{n-1}); terms corresponding to the step after that is replaced with β (in this case y_n and f_n); the terms corresponding to the step after that one is replaced with β^2 (in this case y_{n+1}. Then we obtain

$$\beta^2 = \beta + \Delta t\left[\alpha c\beta + (1-\alpha)c\right] = \beta + \alpha z\beta + (1-\alpha)z$$

$$\beta^2 - (1+\alpha z)\beta - z(1-\alpha) = 0$$

where we substituted $z = c\Delta t$. Obviously, this is just a short cut to getting to the same result.

1.6. Question 27

The correct answer is H. We have

$$\begin{aligned} y_{n+1} &= y_n + \Delta t\left[\frac{3}{4}f(y_n) + \frac{1}{4}f(y_{n+1})\right] \\ y_{n+1} - y_n = \Delta y_n &= \Delta t\left[\frac{3}{4}f(y_n) + \frac{1}{4}f(y_{n+1})\right] \end{aligned}$$

The Taylor series expansion of $f(y_n)$ is simply $f(y_n)$; for $f(y_{n+1})$, you need to take

$$f(y_{n+1}) = f(y_n) + \Delta y f_y(y_n) + O\left(\Delta y^2\right)$$

and thus

$$\begin{aligned} \Delta y_n &= \Delta t\left[\frac{3}{4}f(y_n) + \frac{1}{4}\left[f(y_n) + \Delta y f_y(y_n) + O\left(\Delta y^2\right)\right]\right] \\ \left(1 - \frac{\Delta t}{4}f_y\right)\Delta y_n &= \Delta t f(y_n) + O\left(\Delta t^3\right) \end{aligned}$$

Part II

Quiz 2015

1.7. Question 1

The correct answer is E. There are numerous ways to do this. First, you could just happen to remember the stability formulas for the forward and backward Euler method. Secondly, you can derive the stability formulas here on paper. Thirdly, you can check manually as follows. For the forward-Euler method, we can write

$$y_{n+1} = y_n + \Delta t f(y_n) = y_n + 0.6 f(y_n)$$

With $u' = -\frac{5}{2}u$, this leads to $f(y_n) = -\frac{5}{2}y_n$, and thus

$$y_{n+1} = y_n - 0.6\frac{5}{2}y_n = -\frac{1}{2}y_n$$

Note that this means that the absolute value of y_{n+1} will always be smaller than the previous value y_n. This means that an error in y_n will never propagate to create infinite disturbances. For the backward-Euler method, we can do the same:

$$y_{n+1} = y_n + \Delta t f(y_{n+1}) = y_n + 0.6 f(y_{n+1})$$

where $f(y_{n+1}) = -\frac{5}{2}y_{n+1}$, and thus

$$\begin{aligned} y_{n+1} &= y_n + 0.6 \cdot -\frac{5}{2}y_{n+1} = y_n - \frac{3}{2}y_{n+1} \\ \frac{5}{2}y_{n+1} &= y_n \\ y_{n+1} &= \frac{2y_n}{5} \end{aligned}$$

Again, y_{n+1} will always be smaller than y_n, and thus disturbances won't cause instability. So, both are stable.

Alternatively, using the formula for forward-Euler

$$|1 - 2hc| < 1$$

with $c = f_y(y_n) = -\frac{5}{2}$, we deduce that for $h = 0.6$, this equation is indeed satisfied. The same holds for

$$\left|\frac{1}{1-hc}\right|$$

1.8. Question 3

Answer H is correct. All should be pretty obvious. Only the floating-point rounding errors may seem vague at first: however, in a floating-point system, it's not a good idea to combine very small numbers with very large numbers, as this will cause rounding errors (as you have only a fixed number of digits available to you).

1.9. Question 4

The correct answer is B. Note that we can rewrite it to (as we have 8 steps)

$$u_{i+1} = u_i + \Delta t f(u_{i+1}) = u_i - \frac{c}{2} u_{i+1}$$

$$\begin{aligned}
\left(1+\frac{c}{2}\right)u_{i+1} &= u_i \\
u_{i+1} &= \frac{u_i}{1+\frac{c}{2}} \\
u_{i+n} &= \frac{u_i}{\left(1+\frac{c}{2}\right)^n} \\
u_n &= \frac{u_0}{\left(1+\frac{c}{2}\right)^8} = \frac{1}{\left(1+\frac{c}{2}\right)^8} = \left(1+\frac{c}{2}\right)^{-8}
\end{aligned}$$

1.10. Question 5

The correct answer is C. The Taylor series expansion of u_{i+1} is

$$u_{i+1} = u_i + \Delta t u_i' + \frac{(\Delta t)^2}{2} u_i'' + O(\Delta t)^3 = u_i + \Delta t f_t(u_i) + \frac{(\Delta t)^2}{2} f_{tt}(u_i) + O(\Delta t)^3$$

For $f(u_{i-1})$, it is

$$f(u_{i-1}) = f(u_i) - \Delta t f_t(u_i) + O\left(\Delta t^2\right)$$

Combining them leads to

$$\begin{aligned}
u_i + \Delta t f_t(u_i) + \frac{(\Delta t)^2}{2} f_{tt}(u_i) + O\left(\Delta t^3\right) &= u_i + \frac{3}{2}\Delta t f(u_i) + \frac{1}{2}\Delta t\left(f(u_i) - \Delta t f_t(u_i) + O\left(\Delta t^2\right)\right) \\
\left(1+\frac{\Delta t}{2}\right)\Delta t f_t(u_i) + \frac{(\Delta t)^2}{2} f_{tt}(u_i) + O\left(\Delta t^3\right) &= 2\Delta t f(u_i) + O\left(\Delta t^3\right)
\end{aligned}$$

So, clearly, the smallest order of magnitude at which it goes wrong is $O\left(\Delta t^1\right)$ and thus answer C is correct.

1.11. Question 6

The correct answer is C. The function is decreasing around $h = 0$ ($J'(x_0) < 0$), thus it's a maximum. However, we don't know anything about whether it's global or not; for that, we'd need information on $O\left(h^3\right)$ which we don't have.

1.12. Question 7

The correct answer is E. None of the methods is guaranteed to converge to the global minimum. The other statements are just very true.

1.13. Question 8

The correct answer is C. First, finding the objective function; the surface area is equal to

$$S = l^2 + 4hl$$

with the constraint

$$l^2 h = 10$$

so that $h = 10/l^2$ and thus

$$S = l^2 + \frac{40}{l}$$

Thus, we have

$$\begin{aligned} S' &= 2l - \frac{40}{l^2} \\ S'' &= 2 + \frac{80}{l^3} \end{aligned}$$

and thus

$$l_1 = l_0 - \frac{S'(l_0)}{S''(l_0)} = 1.0 - \frac{2 \cdot 1.0 - \frac{40}{1^2}}{2 + \frac{80}{1^2}} = 1.46$$

1.14. Question 9

The correct answer is C. We have

$$\begin{aligned} f(x) &= 7x - \ln(x) \\ f'(x) &= 7 - \frac{1}{x} \\ f''(x) &= \frac{1}{x^2} \\ f'(0.1) &= 7 - \frac{1}{0.1} = -3 \\ f''(0.1) &= \frac{1}{0.1^2} = 100 \end{aligned}$$

and thus

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 0.1 - \frac{-3}{100} = 0.13$$

However, the exact minimum is located at

$$\begin{aligned} f'(x) = 7 - \frac{1}{x} &= 0 \\ x &= \frac{1}{7} \end{aligned}$$

so that the error is $\epsilon = 0.142857 - 0.13 = 0.0129$.

1.15. Question 10

The correct answer is D. For steepest descent, we need to know the slope in each dimension. We can do that by computing

$$f_x(x, y) \approx \frac{f(x_{i+1}, y_i) - f(x_i, y_i)}{2}$$
$$f_y(x, y) \approx \frac{f(x_i, y_{i+1}) - f(x_i, y_i)}{2}$$

Thus, we need three points: (x_i, y_i), (x_{i+1}, y_i) and (x_i, y_{i+1}).

Part III

2016 quiz

1.16. Question 1

The correct answer is D. It's always one order lower.

1.17. Question 2

The correct answer is B. We have

$$\begin{aligned} z_1' &= z_2 \\ z_2' &= -10\sin(z_1) \end{aligned}$$

Then, we first compute $z_2(t_1)$:

$$z_2(t_0) = z_2(t_0) + \Delta t z_2'(t_0) = 0 + 0.001 \cdot -10 \cdot \sin\left(\frac{\pi}{2}\right) = -0.01$$

Then, for $z_1(t_1)$:

$$z_1(t_1) = z_1(t_0) + \Delta t z_1'(t_0) = \frac{\pi}{2} + 0.001 \cdot 0 = \frac{\pi}{2}$$

and thus answer B is correct.

1.18. Question 3

The correct answer is E. We have

$$y_{n+1} - y_n = \Delta y = \frac{\Delta t}{2}\left[f(y_{n+1}) + f(y_n)\right]$$

The linearisation of $f(y_{n+1})$ is

$$f(y_{n+1}) = f(y_n) + \Delta y f_y(y_n) + \dots$$

This leads to

$$\begin{aligned} \Delta y &= \frac{\Delta t}{2}\left[f(y) + \Delta y f_y(y_n) + f(y_n)\right] \\ \Delta y - \Delta y \cdot \frac{\Delta t}{2} \cdot f_y(y_n) &= \Delta t f(y_n) \\ \Delta y\left(1 - \frac{\Delta t}{2} f_y(y_n)\right) = \Delta t f(y_n) & \\ \Delta y &= \Delta t\left[1 - \frac{\Delta t}{2} f_y(y_n)\right]^{-1} f(y_n) \\ y_{n+1} &= y_n + \Delta t\left[1 - \frac{\Delta t}{2} f_y(y_n)\right]^{-1} f(y_n) \end{aligned}$$

1.19. Question 4

The correct answer is G. We must have

$$|1 - 2h| < 1$$

This is the case for $0 < h < 1$.

1.20. Question 5

The correct answer is B. Remember the formulas

$$\begin{aligned}\eta_{i+1} &= \sum_{k=1}^{s} a_{s-k}\eta_{i+1-k} + z\sum_{k=0}^{s} b_{s-k} f_{i+1-k} \\ \beta^s &= \sum_{k=1}^{s} a_{s-k}\beta^{s-k} + z\sum_{k=0}^{s} b_{s-k}\beta^{s-k}\end{aligned}$$

From comparison, we have $s = 2$ (we need f_{i-1} to appear). Then:

- For $k = 0$:
  - For $b_{2-0} = b_2$, we get f_{i+1}, which doesn't appear in our (rewritten) scheme, thus $b_2 = 0$.
- For $k = 1$:
  - For $a_{2-1} = a_1$, we get η_i, for which the coefficient is simply 1. Thus, $a_1 = 1$.
  - For $b_{2-1} = b_1$, we get f_i, for which the coefficient is $\frac{3}{2}$. Thus, $b_1 = \frac{3}{2}$.
- For $k = 2$:
  - For $a_{2-2} = a_0$, we get η_{i-1}, for which the coefficient is 0 as it does not appear in our formula. Thus, $a_0 = 0$.
  - For $b_{2-2} = b_0$, we get f_{i-1}, for which the coefficient is $-\frac{1}{2}$. Thus, $b_2 = -\frac{1}{2}$.

Plugging this into the second equation yields

$$\begin{aligned}\beta^2 = a_1\beta^1 + z\left(b_1\beta^1 + b_0\beta^0\right) &= \beta + z\left(\frac{3}{2}\beta - \frac{1}{2}\right) \\ \beta^2 - \left(1 + \frac{3}{2}z\right)\beta + \frac{z}{2} &= 0\end{aligned}$$

Alternatively, you can remember that you multiply each derivative with β to the power of the order of the derivative (i.e., ys are simply multiplied with $c^0 = 1$, fs are multiplied with $c^1 = c$, f's are multiplied with c^2, etc.), and that the terms corresponding to 'oldest' step appearing in the scheme is replaced with 1 (in this case f_{i-1}); terms corresponding to the step after that is replaced with β (in this case y_i and f_i); the terms corresponding to the step after that one is replaced with β^2 (in this case y_{i+1}. Then we obtain

$$\beta^2 = \beta + \frac{\Delta t}{2}[3c\beta - c] = \beta + \frac{z}{2}[3\beta - 1]$$
$$\beta^2 - \left(1 + \frac{3}{2}z\right)\beta + \frac{z}{2} = 0$$

where we substituted $z = c\Delta t$. Obviously, this is just a short cut to getting to the same result.

1.21. Question 6

The correct answer is H. All statements are correct. The first one is just a definition, and the second one is just a definition too: the local error is the error that arises when you plug in the exact value of $y(t_i)$ to approximate $y(t_{i+1})$. The last one is quite similar to the degree of precision of a quadrature formula. If $\mathcal{D}$ is able to calculate the derivative of $y(t)$ exactly, then the result will be exact as well.

1.22. Question 7

The correct answer is D. We have

$$\begin{aligned} V &= \pi r^2 h \\ S &= 2\pi r^2 + 2\pi r \cdot h \end{aligned}$$

However, as $V = 311\,\text{cm}^3$, we have $h = 311/(\pi r^2)$, and thus

$$S = 2\pi r^2 + 2\pi r \cdot \frac{311}{\pi r^2} = \frac{622}{r}$$

Minimizing this using your graphical calculator yields $r = 3.67\,\text{mm}$.

1.23. Question 8

The correct answer is F. We simply have

$$\begin{aligned} g(x) &= x^4 - 22x^2 + x + 114 \\ g'(x) &= 4x^3 - 44x + 1 \\ g''(x) &= 12x^2 - 44 \\ g'(2) &= 4 \cdot 2^3 - 44 \cdot 2 + 1 = -55 \\ g''(2) &= 12 \cdot 2^2 - 44 = 4 \end{aligned}$$

so that

$$x_1 = x_0 - \frac{g'(x_0)}{g''(x_0)} = 2 - \frac{-55}{4} = 2 + \frac{55}{4}$$

1.24. Question 9

The correct answer is G. A local minima of the algorithm may also be caused by a local *maxima* below the x-axis (just think about it: a local maxima below the x-axis will also cause a minimum value in $\frac{1}{2}[g(x)]^2$), for which we know that $g''(x_i) > 0$ as it's a local maxima. However, the product $g(\tilde{x}_i) \cdot g''(\tilde{x}_i)$ will stay positive, so that's why answer G is correct.

1.25. Question 10

The correct answer is B. The nodes are ordered x_1, x_3, x_4, x_2. If $f(x_3) > f(x_4)$, then you should be left over with the interval $[x_3, x_2]$. However, this is not the case: the interval used in the next step becomes $[x_1, x_4]$. Correct would have been if line 7 read if $f(x_3) < f(x_4)$.