Part I

Quiz 2013: module 3

1.1. Question 18

The correct answer is E. The following requirements apply:

- We have 7 subintervals; for each it is required that at both end points of the subinterval, $p_i(x_i) = f_i$ and $p_i(x_{i+1}) = f_{i+1}$, for $i = 0, \ldots, 6$. This amounts to two requirements per subinterval, thus $2 \cdot 7$ in total.
- For each of the five derivatives, it is required that at the 6 inner nodes, $p_k^{(n)}(x_{k+1}) = p_{k+1}^{(n)}(x_{k+1})$ with $k = 0, \ldots, 5$ and $n = 1, \ldots, 5$. Thus we have 5 times 6 requirements, amounting to $5 \cdot 6 = 30$ requirements.

In total, this adds up to 44.

1.2. Question 19

The correct answer is E. We are interpolating a 4th order polynomial with a 4th order polynomial, so whatever data set you use, you end up at exactly the function you're interpolating. On the other hand, for the cubic splines, it definitely matters what data set you use.

1.3. Question 20

The correct answer is G. We have the requirements

$$s_0(0) = 0, \quad s_0(1) = 1, \quad s_0''(0) = 0, \quad s_0'(1) = 0$$

where the last requirement comes from symmetry. With

$$\begin{aligned} s_0(x) &= ax^3 + bx^2 + cx + d \\ s_0'(x) &= 3ax^2 + 2bx + c \\ s_0''(x) &= 6ax + 2b \end{aligned}$$

this leads to the system of equations

$$\begin{aligned} s_0(0) &= a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = d = 0 \\ s_0(1) &= a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = a + b + c + d = 1 \\ s_0''(0) &= 6a \cdot 0 + 2b = 2b = 0 \\ s_0'(1) &= 3a \cdot 1^2 + 2b \cdot 1 + c = 3a + 2b + c = 0 \end{aligned}$$

Solving this system with your graphical calculator leads to $a = -1/2$, $c = 3/2$ and $b = d = 0$, so that

$$\begin{aligned} s_0(x) &= -\frac{x^3}{2} + \frac{3x}{2} \\ s_0\left(\frac{1}{2}\right) &= -\left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} + \frac{3 \cdot \frac{1}{2}}{2} = \frac{11}{16} \end{aligned}$$

so answer G is correct.

1.4. Question 21

The correct answer is G. As we have quadratic polynomials, this means that for each of x_1, x_2, x_3,...,x_10, 3 points are needed. This means that you get a grid of $3 \times 3 \times 3 \times \cdots \times 3$, or $3^{10} = 59049$ grid points. If you are slightly confused: suppose we had $f(x, y)$, and we would use tensor-product interpolation using quadratic polynomials: we'd then want x_0, x_1 and x_2, and y_0, y_1 and y_2, so that we'd get 9 grid points (x_0, y_0), (x_0, y_1),..., (x_1, y_0), ..., (x_2, y_2). If we'd add another variable z, we'd be getting $3 \cdot 9 = 27$ data points, as we also need to construct the grid in z-direction.

1.5. Question 22

The correct answer is D. Remember we have to solve

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{f}$$

where the entries of $\mathbf{A}$ are given by $A_{ij} = \Phi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right)$. As we have $\phi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right) = r^2$, we simply have

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix}(x_0-x_0)^2+(y_0-y_0)^2 & (x_0-x_1)^2+(y_0-y_1)^2 & (x_0-x_2)^2+(y_0-y_2)^2\\(x_1-x_0)^2+(y_1-y_0)^2 & (x_1-x_1)^2+(y_1-y_1)^2 & (x_1-x_2)^2+(y_1-y_2)^2\\(x_2-x_0)^2+(y_2-y_0)^2 & (x_2-x_1)^2+(y_2-y_1)^2 & (x_2-x_2)^2+(y_2-y_2)^2\end{bmatrix}\\ &= \begin{bmatrix}0 & (0-1)^2+(0-1)^2 & (0-1)^2+(0-2)^2\\(1-0)^2+(1-0)^2 & 0 & (1-1)^2+(1-2)^2\\(1-0)^2+(2-0)^2 & (1-1)^2+(2-1)^2 & 0\end{bmatrix} = \begin{bmatrix}0 & 2 & 5\\2 & 0 & 1\\5 & 1 & 0\end{bmatrix}\end{aligned}$$

Invert this matrix (with your graphical calculator) and multiply

$$\mathbf{a} = \mathbf{A}^{-1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\-\frac{5}{4}\\\frac{1}{2}\end{bmatrix}$$

so $a_0 = \frac{1}{4}$ and answer D is correct.

1.6. Question 23

The correct answer is B: note that whatever the coefficients are, the basis functions are all 0 at (0,0), as the all the points are located a distance larger than 1/10 from (0,0); essentially, we get

$$\phi(0,0) = a_1\cdot 0 + a_2\cdot 0 + a_3\cdot 0 + a_4\cdot 0$$

1.7. Question 24

The correct answer is E. For E, the interpolant is not linear in a_2. If you recall how we derived everything for least-squares, it was required that a_2 is linear (that means, it's simply $a_2\cdot f(x,y)$, and not a_2 that appears in the midst of a function such as the natural logarithm). This was because we differentiated at some point in time, and differentiating this function (answer E) would result in a really ugly expression which is not linear, and thus you could not solve by matrix algebra.

1.8. Question 25

The correct answer is D. The linear interpolant can easily be found:

$$p = a_0\alpha + a_1\beta + a_2$$

For $i = 0$, we immediately see that $a_2 = 0$. Furthermore, looking at $i = 1$ compared to $i = 0$, we see that $a_0 = \frac{1}{2}$. Then, comparing $i = 1$ and $i = 2$, we see that p should have only increased by $0.5\cdot(120-80) = 20$, but it has in fact increased by 30; so, the increase of β by 5 has increased p by 10: $a_1 = 2$. Thus, in general, we have

$$p = \frac{1}{2}\alpha + 2\beta$$

If we have $p = 65$, this leads to

$$\begin{aligned} 65 &= \frac{1}{2}\alpha + 2\beta \\ \alpha &= 130 - 4\beta \end{aligned}$$

so answer D is correct.

Part II

Quiz 2013: module 4

1.9. Question 20

The correct answer is C. First, consider $f(x) = \sin(x)$ with $h = \pi/2$:

$$\begin{aligned} D_F(\sin(x),0) &= \frac{f(0+\pi/2)-f(0)}{\pi/2} = \frac{1-0}{\pi/2} = \frac{2}{\pi} \\ D_B(\sin(x),0) &= \frac{f(0)-f(0-\pi/2)}{\pi/2} = \frac{0--1}{\pi/2} = \frac{2}{\pi} \end{aligned}$$

so that $D_F(f,0) = D_B(f,0)$. For $g(x) = x^2$, let $h = 1$, then we get:

$$\begin{aligned} D_F\left(x^2,x\right) &= \frac{f(0+1)-f(0)}{1} = \frac{1-0}{1} = 1 \\ D_B\left(x^2,x\right) &= \frac{f(0)-f(0-1)}{1} = \frac{0-1}{1} = -1 \end{aligned}$$

so that $D_F(g,0) = -D_B(g,0)$ and thus answer C is correct.

1.10. Question 21

The correct answer is D. If you draw a sketch of $f(x) = e^x$, which is an monotonously increasing function (i.e. the slope is always larger than 0), and draw a certain point somewhere on the graph, then another one forward of this point, then you quickly realize that always the slope of the straight line connecting these points will always be larger than the slope of e^x at the first point, and thus $\epsilon \equiv f'(x) - D_F(f,x)$ will always be negative.

1.11. Question 22

The correct answer is B. You can again draw it, but it's probably not immediately clear then. However, you can prove that the error is actually equal to zero:

$$D_C\left(x^2,x\right) = \frac{(x+h)^2-(x-h)^2}{2h} = \frac{x^2+2hx+h^2-x^2+2hx-h^2}{2h} = \frac{4hx}{2h} = 2x$$

and since $f'(x) = 2x$ as well, we simply have

$$\epsilon = 2x - 2x = 0$$

1.12. Question 23

The correct answer is B. What we will do is that we will expand each of the functions as Taylor series: the terms relating to $f(x)$ should then cancel out; the terms relating to $f'(x)$ reduce to exactly $f'(x)$ (when divided by $6h$); if either of these are not satisfied, we are not approximating $f'(x)$ to begin with. The Taylor expansions can be written as

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2}f''(x) + O\left(h^3\right) = f(x) + 2hf'(x) + 2h^2f''(x) + O\left(h^3\right) \quad (1.1)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O\left(h^3\right) \quad (1.2)$$

$$f(x) = f(x) \quad (1.3)$$

such that when plugging it in in the given scheme, we obtain

$$
\begin{aligned}
f'(x) &= \frac{2f(x+2h)-f(x+h)-f(x)}{6h}+\epsilon \\
&= \frac{2f(x)+4hf'(x)+2h^2f''(x)+O(h^3)-f(x)-hf'(x)-\frac{h^2}{2}f''(x)+O\left(h^3\right)-f(x)}{6h}+\epsilon \\
&= \frac{3hf'(x)+\frac{3}{2}h^2f''(x)+O\left(h^3\right)}{6h}+\epsilon \\
&= \frac{1}{2}f'(x)+\frac{1}{4}hf''(x)+O\left(h^2\right)+\epsilon
\end{aligned}
$$

Thus, this rule is inconsistent, as the right-hand side contains $\frac{1}{2}f'(x)$ instead of just $f'(x)$. Thus, answer B is correct. Note: in case you *would* have obtained $f'(x)$ on the right-hand-side, you should look at the term with the smallest order h in there. In this case, that would be $hf''(x)/4$, so the error would be have like $O(h)$. However, it might have happened that the $f''(x)$ terms cancelled out, and then you'd have to analyse the $f'''(x)$ terms and those would have had a h^2 term in front of them (after dividing by $6h$), so it would become $O(h^2)$.

1.13. Question 24

The correct answer is D. We change the interval from $[-1,1]$ to $[0,2]$. We have

$$x_i' = \frac{d-c}{b-a}(x_i-a)+c = \frac{2-0}{1--1}(x_i--1)+0 = x_i+1$$

so that

$$
\begin{aligned}
x_1' &= -\frac{1}{\sqrt{3}}+1 = 1-\frac{1}{\sqrt{3}} \\
x_2' &= \frac{1}{\sqrt{3}}+1 = 1+\frac{1}{\sqrt{3}}
\end{aligned}
$$

so that

$$Q(f,[0,2]) = f\left(1-\frac{1}{\sqrt{3}}\right)+f\left(1+\frac{1}{\sqrt{3}}\right) = \left(1-\frac{1}{\sqrt{3}}\right)^3+\left(1+\frac{1}{\sqrt{3}}\right)^3 = 4$$

and thus answer D is correct.

1.14. Question 25

The correct answer is E. Either you remember this is Simpson's rule which is known to have degree of precision 3, or you just manually check, first using $f(x)=1$, $f(x)=x$, $f(x)=x^2$, and so on. I assume you're capable enough of doing the first two yourself, but I'll show you how it goes from there on:

$$
\begin{aligned}
\int_0^2 x^2 dx &= \frac{8}{3} \\
Q\left(x^2\right) = \frac{1}{3}\cdot 0^2+\frac{4}{3}\cdot 1^2+\frac{1}{3}\cdot 2^2 = \frac{8}{3}
\end{aligned}
$$

so the degree of precision is at least 2. Then:

$$\int_0^2 x^3 dx = 4$$
$$Q\left(x^3\right) = \frac{1}{3}\cdot 0^3 + \frac{4}{3}\cdot 1^3 + \frac{1}{3}\cdot 2^3 = 4$$

so the degree of precision is at least 3. Then:

$$\int_0^2 x^4 dx = \frac{32}{5}$$
$$Q\left(x^4\right) = \frac{1}{3}\cdot 0^4 + \frac{4}{3}\cdot 1^4 + \frac{1}{3}\cdot 2^4 = \frac{20}{3}$$

so the degree of precision is 3 and thus answer E is correct.

1.15. Question 26

The correct answer is G. The quickest way to solve this question is by thinking, symmetry is usually good, so let's make it symmetric about the center of the interval, so that $x_1 = 2/3$. Alternatively, we take the long way home. First, checking $f(x) = 1$:

$$\int_0^1 dx = 1 = \frac{1}{2}\cdot 1 + w_1 \cdot 1$$

so we must have $w_1 = 1/2$ (this also follows from the fact the sum of the weights should equal the length of the interval). Then, for $f(x) = x$:

$$\int_0^1 x dx = \frac{1}{2} = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot x_1$$

Solving this yields $x_2 = 1/3$ and thus answer G is correct. Note that we don't have to continue any longer, as x_1 is fixed now: any value different from this will cause the degree of precision to be 0, as it's no longer exact for $d = 1$.

1.16. Question 27

The correct answer C: we have 6 nodes, and thus the interpolating polynomial is able to exactly represent a polynomial of degree 3 (for which only 4 nodes are necessary). Thus, the Lagrange polynomial is identical to $f(x)$. However, the natural cubic spline will not be identical to $f(x)$: the second derivatives at the end point of $x_5 = 1$ is not generally equal to 0, so the function will be definitely different in that part of the function.

Part III

Quiz 2015

1.17. Question 1

The correct answer is D. This question requires careful reading: $\phi(x)$ is cubic, so $\phi'' \in C^2([a,b])$. This means that ϕ' is only $\phi' \in C^1([a,b])$. If you didn't see the ϕ' but read ϕ, you'd be wrong.

1.18. Question 2

The correct answer is G. Polynomial interpolation will give you the exact same function, as you interpolate at three nodes and thus have a quadratic interpolating polynomial. Interpolating a quadratic polynomial with a quadratic polynomial leads to the exact same result. Linear spline interpolation obviously leads to something different, as the subintervals are interpolated by linear functions, which obviously is different. Cubic spline interpolation with natural BCs will also be different, as the second derivative at x_0 and x_2 is not zero. Cubic spline interpolation with clamped BCs *will* lead to the same function: the derivatives at the end points are the same, and the cubics will simply become quadratic splines. So, a and d lead to the correct function and thus answer G is correct.

1.19. Question 3

The correct answer is F. You may be inclined to calculate all of c_0, d_0, a_1, b_1, c_1 and d_1, but this is actually not even necessary. We have $S_1''(1) = 0$ and

$$\begin{aligned} S_0(x) &= -\frac{1}{2}(x+1)^3 + c_0(x+1) + d_0 \\ S_0''(x) &= -3(x+1) \\ S_0''(0) &= -3(0+1) = -3 \end{aligned}$$

Thus, as we have

$$S_1''(x) = 6a_1x + 2b_1$$

we have two equations ($S_1''(0) = -3$ and $S_1''(1) = 0$), this leads to the following system of equations:

$$\begin{aligned} S_1''(0) &= 6a_1 \cdot 0 + 2b_1 = 2b_1 = -3 \\ S_1''(1) &= 6a_1 \cdot 1 + 2b_1 = 6a_1 + 2b_1 = 0 \end{aligned}$$

and thus we have $b_1 = -3/2$ and $a_1 = -1/3b_1 = 1/2$ and thus answer F is correct.

1.20. Question 4

The correct answer is B. Essentially, our interpolating polynomial will look like

$$\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4\right)\left(b_0 + b_1y + b_2y^2\right)$$

as we have five nodes in x-direction and three nodes in y-direction. Clearly, each of the given monomials appears in this: a_1xb_1y becomes c_4xy for example, $a_4x^4b_2y^2$ becomes $c_{14}x^4y^2$, $a_2x^2b_2y^2 = c_7x^2y^2$, etc.

1.21. Question 5

The correct answer is H. Seems much harder than it really is. We have the requirement that

$$\phi(0) = a_0\phi_0(0) + a_1\phi_1(0) + a_2\phi_2(0) = a_0 \cdot \frac{1}{10}|0+1| + a_1 \cdot |0| - \frac{1}{40} \cdot 10|0-1| = 0$$

or

$$\frac{a_0}{10} - \frac{1}{4} = 0$$

and thus $a_0 = 10/4 = 5/2$ and thus answer H is correct.

1.22. Question 6

The correct answer is F. Even without using the hint, you can see it yourself: it rather clearly is the magnitude of the difference in slopes between two subsequent intervals. This obviously has to do with the second derivative. However, for it to be precisely $f''(x_i)$, one would need to divide by the length of the interval, i.e. h. Therefore, we're actually computing $hf''(x_i)$ rather than $f''(x_i)$ itself. Alternatively, you can use Taylor series for this:

$$\begin{aligned} f(x_i - h) &= f(x_i) - hf'(x_i) + \frac{h^2}{2} f''(x_i) + O\left(h^3\right) \\ f(x_i + h) &= f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + O\left(h^3\right) \end{aligned}$$

so that we can write

$$\begin{aligned} \frac{f(x_i+h) - f(x_i)}{h} - \frac{f(x_i) - f(x_i - h)}{h} &= \frac{f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + O\left(h^3\right) - f(x_i)}{h} \\ &\quad - \frac{f(x) - f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + O\left(h^3\right)}{h} \\ &= \frac{hf'(x_i) + \frac{h^2}{2} f''(x_i) + O\left(h^3\right)}{h} - \frac{hf'(x_i) + \frac{h^2}{2} f''(x_i) + O\left(h^3\right)}{h} \\ &= f'(x_i) + \frac{h}{2} f''(x_i) - f'(x_i) + \frac{h}{2} f''(x_i) + O\left(h^2\right) = hf''(x_i) + O\left(h^2\right) \end{aligned}$$

Please note that I can actually recommend doing the Taylor series for this one; you may have confused yourself by thinking that considering we have the magnitude of the difference in slopes between two subsequent intervals, we would need to divide by $2h$, but clearly we don't.

1.23. Question 7

The correct answer is F. For the Lagrange basis, we had

$$p(x) = f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x)$$

where

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

In this case, $n = 2$. We then have

$$\begin{aligned} l_0(x) &= \prod_{j=0, j\neq 0}^{2} \frac{x - x_j}{-h - x_j} = \frac{x-0}{-h-0} \frac{x-h}{-h-h} = \frac{x^2 - xh}{2h^2} \\ l_1(x) &= \prod_{j=0, j\neq 1}^{2} \frac{x - x_j}{0 - x_j} = \frac{x - -h}{0 - -h} \frac{x-h}{0-h} = \frac{x^2 - h^2}{-h^2} \\ l_2(x) &= \prod_{j=0, j\neq 2}^{2} \frac{x - x_j}{h - x_j} = \frac{x - -h}{h - -h} \frac{x-0}{h-0} = \frac{x^2 + xh}{2h^2} \end{aligned}$$

so that

$$\begin{aligned} p(x) &= f_0\frac{x^2-xh}{2h^2}-f_1\frac{x^2-h^2}{h^2}+f_2\frac{x^2+xh}{2h^2} \\ p'(x) &= f_0\frac{2x-h}{2h^2}-f_1\frac{2x}{h^2}+f_2\frac{2x+h}{2h^2}=\frac{1}{h^2}\left[\frac{2xf_0-hf_0}{2}-\frac{2xf_1}{1}+\frac{2xf_2+hf_2}{2}\right] \\ &= \frac{1}{2h^2}\left[f_0(2x-h)-4xf_1+f_2(2x+h)\right] \end{aligned}$$

so that

$$\begin{aligned} p'\left(\frac{h}{3}\right) &= \frac{1}{2h^2}\left[f_0\left(2\cdot\frac{h}{3}-h\right)-4\cdot\frac{h}{3}\cdot f_1+f_2\cdot\left(2\cdot\frac{h}{3}+h\right)\right]=\frac{1}{2h^2}\left(f_0\cdot\frac{-h}{3}-4\cdot\frac{h}{3}f_1+5f_2\cdot\frac{h}{3}\right] \\ &= \frac{1}{6h}\left[-f_0-4f_1+5f_2\right] \end{aligned}$$

and thus answer F is correct.

1.24. Question 8

The correct answer is D. Just remember the formula

$$w_i'=\frac{d-c}{b-a}w_i=\frac{h}{1--1}w_i=\frac{h}{2}w_i$$

so that

$$\begin{aligned} w_1' &= \frac{h}{2}\cdot\frac{4}{3}=\frac{4h}{6}=\frac{2h}{3} \\ w_2' &= \frac{h}{2}\cdot\frac{1}{3}=\frac{h}{6} \end{aligned}$$

so answer D is correct.

1.25. Question 9

The correct answer is D. The derivative of

$$\phi(x_0,y_0)\equiv\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

with respect to y will be

$$\begin{aligned} \frac{\partial\phi}{\partial y} &= \frac{\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)}{\Delta x}-\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}}{\Delta y}=\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)-f(x_0+\Delta x,y_0)+f(x_0,y_0)}{\Delta x\Delta y} \\ &= \frac{f_{11}-f_{01}-f_{10}+f_{00}}{\Delta x\Delta y} \end{aligned}$$

and thus answer D is correct.

1.26. Question 10

The correct answer is G. We have

$$E(f;x) = \frac{1}{(2+1)!}(x--h)x(x-h) = \frac{1}{6}\left|\left(x^3 - xh^2\right)\right|$$

Thus, we can compute the maximum value of error at each point x, bearing in mind those absolute value thingies. Integrating this will simply give the error for the interval on the interval $[-h,h]$. However, once again, we must bear the absolute value thingies in mind: on the interval $[-h,0]$, the thing between brackets is positive, whereas between $[0,h]$, it's negative. Furthermore, it is point symmetric around the origin (you make your life easier here by just taking $h=1$ and plotting the function on your calculator to visualize it a bit), so the integral becomes

$$\int_{-h}^{h} \frac{1}{6}\left|\left(x^3 - xh^2\right)\right| dx = 2 \cdot \int_{-h}^{0} \frac{1}{6}\left(x^3 - xh^2\right) dx = \frac{1}{3} \cdot \left[\frac{x^4}{4} - \frac{x^2h^2}{2}\right]_{-h}^{0} = \frac{1}{3} \cdot \frac{h^4}{4} = \frac{h^4}{12}$$

and thus answer G is correct.

Part IV

Quiz 2016

1.27. Question 1

The correct answer is F. You could determine a, b, c and d relatively easy. From the requirement $s_0'(0) = s_1'(0)$, we find

$$\begin{aligned} s_0'(x) &= 3ax^2 + 2x + c \\ s_1'(x) &= 3bx^2 - 2x + d \\ s_0'(0) &= s_1'(0) \\ 3a \cdot 0^2 + 2 \cdot 0 + c &= 3b \cdot 0^2 - 2 \cdot 0 + d \end{aligned}$$

and thus $c = d$. Furthermore, from the natural spline, we can deduce a and b:

$$\begin{aligned} s_0''(x) &= 6ax + 2 \\ s_1''(x) &= 6bx - 2 \\ s_0''(-1) &= 6a \cdot -1 + 2 = 2 - 6a = 0 \\ s_1''(2) &= 6b \cdot 2 - 2 = 12b - 2 = 0 \end{aligned}$$

so that $a = \frac{1}{3}$, $b = \frac{1}{6}$. However, then checking that $s_0''(0) = s_1''(0)$, we get

$$s_0''(0) = s_1''(0) \quad (1.4)$$

$$6 \cdot \frac{1}{6} \cdot 0^2 + 2 = 6 \cdot \frac{2}{3} \cdot 0^2 - 2 \quad (1.5)$$

which leads to $2 = -2$ so this is wrong, apparently. Indeed, the main part of this question (and you could have skipped everything so far) is that for whatever values a, b, c and d are, the second derivatives at 0 are not equal to each other, you'll always end up at the inequality

$$2 \neq -2$$

Thus, $s(x)$ is not a spline, and even if we had used different boundary conditions other than the natural boundary conditions (which may have generated different results for a and b), it would not have been a spline. Thus, the correct answer is F.

1.28. Question 2

The correct answer is B. $y(t)$ is the distance, so acceleration is the second derivative. For a natural cubic spline, the second derivatives are zero at the end points of the interval, thus $g = 0\,\text{m/s}^2$ and answer B is correct.

1.29. Question 3

The correct answer is C. Remember that the shape functions are given by

$$\mathbf{S} = \mathbf{b}^T \mathbf{A}^{-1}$$

where $\mathbf{b}^T$ is the vector containing the basis functions and $\mathbf{A}^{-1}$ the inverse of the matrix that follows from the equations that need to be satisfied for the interpolant. For a linear interpolator, we can only use linear functions, i.e.

$$p(x, y) = a_1 + a_2 x + a_3 y$$

meaning that

$$\mathbf{b}^T = \begin{bmatrix} 1 & x & y \end{bmatrix}$$

meaning that answers D and E are definitely wrong. Using this linear interpolant, we get, by plugging in the given vertices:

$$\begin{aligned} p(x_1,y_1) &= p(1,1) = a_1 + a_2\cdot 1 + a_3\cdot 1 = a_1 + a_2 + a_3 = f_1 \\ p(x_2,y_2) &= p(1,3) = a_1 + a_2\cdot 1 + a_3\cdot 3 = a_1 + a_2 + 3a_3 = f_2 \\ p(x_3,y_2) &= p(3,1) = a_1 + a_2\cdot 3 + a_3\cdot 1 = a_1 + 3a_2 + a_3 = f_3 \end{aligned}$$

which can be written as

$$\begin{aligned} \mathbf{Aa} &= \mathbf{f} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} &= \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \end{aligned}$$

so that

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -0.5 & -0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & 0.5 & 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

and thus

$$\mathbf{S} = \mathbf{b}^T\mathbf{A}^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & x & y \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

and thus answer C is correct.

1.30. Question 4

The correct answer is D. Let's just discuss each statement one-by-one:

(i) If we use a polynomial, then it's infinitely continuously differentiable, i.e. $f(x) \in C^{(\infty)}([a,b])$, which is the definition of a smooth function (a function is said to be smooth if it's infinitely many times continuously differentiable).

(ii) Yes this is simply true; in the bivariate case, it's only the case for certain radial basis functions and for certain locations of the data points.

(iii) Yes, that's the entire point of least-squares regression.

(iv) No, it doesn't, unless we have M basis functions with $N = M$ data points (if you have more basis functions than data points, you end up with a non-uniquely solvable system). However, this is generally speaking not the case.

1.31. Question 5

The correct answer is B. If we use a parabola, i.e. $r(x) = ax^2 + bx + c$, then we are essentially interpolating these three nodes. Thus, the parabola passes through the data points, and thus the errors are zero, so answer B is correct.

1.32. Question 6

The correct answer is E. The more basis functions, the closer to interpolating you are; and in fact, as we have 5 nodes and 5 basis functions for (iv), we're already interpolating at (iv), thus that one minimizes the least-squares residual (as the residuals are simply all equal to zero.

1.33. Question 7

The correct answer is C. You can do it quickly by just sketching a function for which the slope is ever-increasing (e.g. e^x), and then picking a point x_0; you can the draw a line connecting to another point forward of x_0 and another line connecting to another point backward of x_0; you'll quickly see that the slope of the forward difference line is larger than the actual slope at x_0, and the slope of the backward difference line is smaller than the actual slope at x_0. This means that for the forward difference, $\epsilon > 0$, but for the backward, $\epsilon < 0$. Thus, answer C is correct. Alternatively, you can do it using Taylor series. For the forward difference one, we have

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(\xi) \\ f'(x) &= \frac{f(x+h)-f(x)}{h} - \frac{h}{2}f''(\xi) \end{aligned}$$

The forward difference scheme simply uses

$$D_f \approx \frac{f(x+h)-f(x)}{h}$$

Thus, if $f''(\xi) > 0$ for all $\xi \in \mathbb{R}$, then we know for sure that $D_f > f'(x)$ and thus $\epsilon > 0$. Similarly, for the backwards scheme, we get

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(\xi) \tag{1.6}$$

$$f'(x) = \frac{f(x)-f(x-h)}{h} + \frac{h}{2}f''(\xi) \tag{1.7}$$

The backward difference scheme simply uses

$$D_b \approx \frac{f(x)-f(x-h)}{h}$$

so that if $f''(\xi) > 0$ for all $\xi \in \mathbb{R}$, then we know for sure that $D_F < f'(x)$.

1.34. Question 8

The correct answer is D. Remember that the Taylor series can be written as

$$f(x_i + kh) = \ldots + \frac{(kh)^5}{5!}f^{(5)}(x_i) + \ldots$$

Furthermore, using the appropriate coefficients, this means that we get that the term involving $f^{(5)}(x_i)$ becomes

$$\frac{1}{12h}\cdot\frac{(-2h)^5}{5!}f^{(5)}(x_i) - \frac{8}{12h}\cdot\frac{(-h)^5}{5!}f^{(5)}(x_i) + \frac{8}{12h}\cdot\frac{h^5}{5!}f^{(5)}(x_i) - \frac{1}{12h}\cdot\frac{(2h)^5}{5!}f^{(5)}(x_i) = -\frac{h^4 f^{(5)}(x_i)}{30}$$

1.35. Question 9

The correct answer is B. Either just remember Simpson's rule, or derive it here, using $x_0 = -1$, $x_1 = 0$ and $x_1 = 1$:

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \int_{-1}^{1} dx \\ \int_{-1}^{1} x dx \\ \int_{-1}^{1} x^2 dx \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$$

and thus we use

$$I \approx \frac{1}{3} \cdot f(-1) + \frac{4}{3} \cdot f(0) + \frac{1}{3} \cdot f(1) = \frac{1}{3} \cdot \left(e^{-1} + 1\right) + \frac{4}{3} \cdot \left(e^{0} + 1\right) + \frac{1}{3} \cdot \left(e^{1} + 1\right) = 4.362$$

Using your graphical calculator to compute the exact integral,

$$I = \int_{-1}^{1} \left(e^{x} + 1\right) dx = 4.350$$

we see that the error is equal to $\epsilon = 4.362 - 4.350 = 0.012$.

1.36. Question 10

The correct answer is C. It's actually not so difficult as the word "Fourier" may make it seem. The solution simply becomes

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \sin(x_0) & \sin(x_1) & \sin(x_2) \\ \cos(x_0) & \cos(x_1) & \cos(x_2) \end{bmatrix}^{-1} \begin{bmatrix} \int_0^{\pi} 1 dx \\ \int_0^{\pi} \sin(x) dx \\ \int_0^{\pi} \cos(x) dx \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \sin(0) & \sin\left(\frac{\pi}{2}\right) & \sin(\pi) \\ \cos(0) & \cos\left(\frac{\pi}{2}\right) & \cos(\pi) \end{bmatrix}^{-1} \begin{bmatrix} \int_0^{\pi} 1 dx \\ \int_0^{\pi} \sin(x) dx \\ \int_0^{\pi} \cos(x) dx \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} - 1 \\ 2 \\ \frac{\pi}{2} - 1 \end{bmatrix}$$

So, note what we actually do: in the first row, we plug in the first function we want to integrate exactly; in the second row, we plug in the second function we want to integrate exactly and in the third row, we plug in the third function we want to integrate exactly; this is very similar to the polynomials we did, where we used the first row for $f(x) = 1$, the second row for $f(x) = x$, the third row for $f(x) = x^2$, and so on.