Part I

Quiz 2013: module 1

1.1. Question 17

Answer C is correct. We have that the Taylor series equals

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

with $x_0=0$. We then have

$$\begin{aligned} f(x) &= e^x &\rightarrow&\quad f(0)=e^0=1\\ f'(x) &= e^x &\rightarrow&\quad f'(0)=e^0=1\\ f''(x) &= e^x &\rightarrow&\quad f''(0)=e^0=1 \end{aligned}$$

and thus

$$f(x)=\frac{1}{0!}(x-0)^0+\frac{1}{1!}(x-0)^1+\frac{1}{2!}(x-0)^2=1+x+\frac{x^2}{2}$$

and thus

$$f(1)=1+1+\frac{1^2}{2}=2.5$$

and thus answer C is correct.

1.2. Question 18

Answer E is correct. From $1\leq s\leq 10-1\times10^{-15}$, it is clear that the step size is 1×10^{-15} and thus the machine epsilon is also 1×10^{-15}.

1.3. Question 19

Answer F is correct. The interval width is halved each time. The interval width after n iterations is thus

$$\frac{1}{2^n}$$

and we must have that this is smaller than $1\cdot10^{-300}$:

$$\begin{aligned} \frac{1}{2^n} &< 1\cdot10^{-300}\\ 2^n &> 10^{300} \end{aligned}$$

Solving this is rather annoying as you'll get overflow errors, so you have to apply a small trick: $10=2^{3.3219}$, and thus

$$2^n>\left(2^{3.219}\right)^{300}=2^{996.58}$$

and thus $n=996$, and thus answer F is correct.

1.4. Question 20

The correct answer is B. We have for Newton's method that

$$\epsilon_{i+1} = \frac{\phi''(\tilde{x})}{2}\epsilon_i^2$$

so that each time, the error is reduced by each square. If $\epsilon_0 = 0.1$, this means that $\epsilon_1 = 0.01$, $\epsilon_2 = 10^{-4}$, $\epsilon_3 = 10^{-8}$, etc., so that we have

$$\epsilon_i = 10^{-2^i} < 10^{-300}$$

and thus $i = 9$, and thus answer B is correct.

1.5. Question 21

The correct answer is C. A is just plain wrong, B is just plain wrong, D is just plain wrong and thus E as well. C is correct, because for FPI, it is required that $\left|\phi'(x)\right| < 1$, but the singularity means that the derivative does not exist at $x_0 = 0$. Therefore, answer C is correct.

1.6. Question 22

The correct answer is A. First, rewrite the function to

$$(x+2)(x-1)(x-3) = (x+2)\left(x^2-4x+3\right) = x^3-4x^2+3x+2x^2-8x+6 = x^3-2x^2-5x+6$$

We know that the roots are located at $x = -2$, $x = 1$ and $x = 3$, and that $x = 0$, $y = 6$ (so it's above the x-axis there, allowing us to make a sketch as shown in figure 1.1.

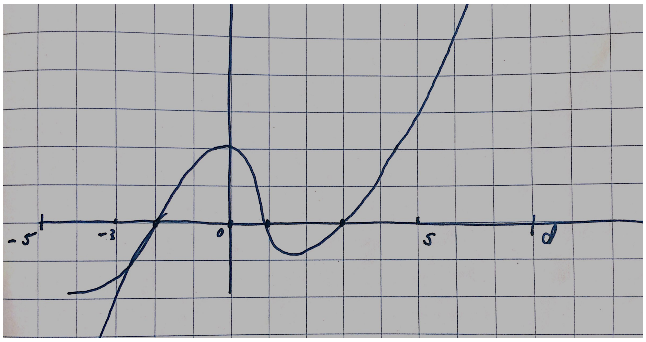

Figure 1.1: Sketch of graph.

In horizontal direction, one cell is equal to one unit x. For 1, $[-1,2]$, it is then quite obvious that it'll converge to $x = 1$. For 2, $[0,8]$, we see that we have positive values for y for both end points, meaning our method fails (and it does not converge to anything, it's just broken). For 3, $[-3,0]$, we quite clearly see that it will converge to $x = -2$ as $x = 1$ is not even in the interval. For 4, $[-5,5]$, we see that the first guess, $x = 0$, will result in a new interval, $[-5,0]$, and thus it'll converge to $x = -2$. Therefore, answer A is correct.

1.7. Question 23

The correct answer is C. We simply move x_{i+1} to the other side to get:

$$\begin{aligned} x_{i+1} &= x_i^2 - 2x_i + 1 \\ 0 &= x_i^2 - 3x_i + 1 \end{aligned}$$

and thus answer C is correct.

1.8. Question 24

The correct answer is A. Remember the formula

$$\begin{aligned} \phi(x) &= x - \frac{f(x)}{f'(x)} \\ f(x) &= 2^x - 1 \\ f'(x) &= \ln(2) \cdot 2^x \\ \phi(x) &= x - \frac{2^x - 1}{\ln(2) \cdot 2^x} \\ \phi(1) &= 1 - \frac{2^1 - 1}{\ln(2) \cdot 2^1} = 0.2787 \end{aligned}$$

and thus answer A is correct.

Part II

Quiz 2013: module 2

1.1. Question 17

The correct answer is E. Remember the relation

$$x_{n+1-i} = \frac{b+a}{2} + \frac{b-a}{2}\xi_i, \quad i = 1, \ldots, n+1$$

x_0 is given to us, i.e. $i = n+1$ and thus we know ξ_{n+1}:

$$\begin{aligned} 4.0170 &= \frac{5+4}{2} + \frac{5-4}{2}\xi_{n+1} \\ \xi_{n=1} &= -0.966 \end{aligned}$$

Then, we can plug in the given equation (note that we must have used a Chebychev polynomial of order $p = n+1$: we have n nodes, thus we require $n+1$ zeros, and thus a polynomial of order $p = n+1$).
Thus:

$$\begin{aligned} \xi_{n+1} &= \cos\left(\frac{2(n+1)-1}{2(n+1)}\pi\right) = -0.966 \\ \frac{2n+1}{2n+2}\pi &= 2.88008 \\ 2n+1 &= 1.8335n + 1.8335 \\ n &\approx 5 \end{aligned}$$

So answer E is correct.

1.2. Question 18

The correct answer is D. First, let's find the second order polynomial: let's use Newton's basis because that's the easiest to do by hand. We have

$$p(x) = d_0 + d_1(x - x_0) + d_2(x - x_0)(x - x_1) = d_0 + d_1 x + d_2 x(x-1)$$

where we can solve d_0, d_1 and d_2 by solving the matrix equation

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} &= \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & (1-0) & 0 \\ 1 & (2-0) & (2-0)(2-1) \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} &= \begin{bmatrix} 0^3 \\ 1^3 \\ 2^3 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} \end{aligned}$$

From the first row, clearly $d_0 = 0$. From the second row then simply $d_1 = 1$. From the third row, we then have $1 \cdot 0 + 2 \cdot 1 + 2 \cdot d_2 = 8$ and thus $d_2 = 3$. We thus get

$$p_2(x) = x + 3x(x-1) = x + 3x^2 - 3x = 3x^2 - 2x$$

which you can verify passes through the required points. We then have

$$\epsilon = \int_0^1 \left|p_2(x) - f(x)\right| dx = \int_0^1 \left|3x^2 - 2x - x^3\right| dx$$

Unfortunately, the absolute value bars do not disappear nicely. Instead, we must find the roots manually to see whether there's a change of sign somewhere between 0 and 1:

$$\begin{aligned} 3x^2-2x-x^3 &= 0 \\ x^3-3x^2+2x &= 0 \\ x\left(x^2-3x+2\right) &= 0 \\ x(x-2)(x-1) &= 0 \end{aligned}$$

Thus, the roots are $x=0$, $x=2$ and $x=1$, meaning there is no change of sign within the interval and we can just integrate between 0 and 1, taking the absolute value of the final answer:

$$\epsilon = \int_0^1 \left|3x^2-2x-x^3\right| dx = \left|\left[x^3-x^2-\frac{x^4}{4}\right]_0^1\right| = \left|-\frac{1}{4}\right| = \frac{1}{4}$$

and answer D is correct.

1.3. Question 19

The correct answer is C. We have that

$$f^{(n+1)}(\xi) = f''' = 3\cdot 2\cdot 1 = 6$$

and $(n+1)! = 3! = 6$. Furthermore,

$$\omega_{n+1}(x) = (x-x_0)(x-x_1)(x-x_2) = x(x-1)(x-2) = x^3-3x^2+2x$$

and thus

$$R_2(f;x) = \frac{6}{6}\left(x^3-3x^2+2x\right) = x^3-3x^2+2x$$

The extreme value occurs when the derivative is zero:

$$\begin{aligned} \frac{d\left(x^3-3x^2+2x\right)}{dx} &= 3x^2-6x+2=0 \\ x^2-2x+\frac{2}{3} &= 0 \\ (x-1)^2-\frac{1}{3} &= 0 \\ x-1 &= \sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3} \\ x &= 1\pm\frac{\sqrt{3}}{3} \end{aligned}$$

Within the interval $[0,1]$, the extreme is thus located at $x = 1-\sqrt{3}/3$ for which

$$R_2\left(f;1-\sqrt{3}/3\right) \approx 0.3849$$

(just plug in your calculator). Now, we don't actually know whether this is an maximum or minimum, so we must also compute $R_2(f;0)=0$ and $R_2(f;1)=0$ and thus we know it is the maximum. 0.3849 corresponds to $2\sqrt{3}/9$, thus answer C is correct.

1.4. Question 20

The correct answer is B. Weierstrass theorem is *always* true (as long as the function is continuous). Answer C is incorrect because it's just bullshit. Answer D may seem very correct to you; however, Cauchy interpolation error theorem only holds for functions that are $n+1$ times differentiable and then still continuous. Answer E is incorrect because the entire point of the Chebychev grid is that it makes the maximum interpolation error go to zero if you increase N.

1.5. Question 21

The correct answer is F. The approximate function is very simply

$$f(t) = 300 - \frac{10t^2}{2} = 300 - 5t^2$$

Solving:

$$\begin{aligned} f(t) = 300 - 5t^2 &= 0 \\ t &\approx 7.75\,\mathrm{s} \end{aligned}$$

1.6. Question 22

The correct answer is F. Set up four interpolation points amounts to setting $f(0) = 300$, $f'(0) = 0$, $f''(0) = -10$ and $f'(8) = -60.8\,\mathrm{m/s}$. Plugging in the first one yields:

$$f(0) = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = 300$$

and thus $d = 300$. The second leads to

$$f'(0) = 3a \cdot 0^2 + 2 \cdot 0 + c = 0$$

and thus $c = 0$. We thus have so far

$$f(t) = at^3 + bt^2 + 300$$

Then, setting up the other two equations:

$$f''(0) = 6a \cdot 0 + 2b = -10$$

and thus $b = -5$, leading to

$$f(t) = at^3 - 5t^2 + 300$$

We have for $t = 8$:

$$\begin{aligned} f'(t) &= 3at^2 - 2 \cdot 5 \cdot t = 3at^2 - 10t \\ f'(8) &= 3a \cdot 8^2 - 10 \cdot 8 = -60.8 \\ 192a &= 19.2 \\ a &= 0.1 \end{aligned}$$

1.7. Question 23

The correct answer is H. It is vital that you read the question carefully: the initial height, at $t = 0$, is given to be precisely 300 m. As

$$\ln\left(\cosh\left(\frac{0}{8}\right)\right) = \ln\left(\cosh\left(\frac{0}{8}\right)\right) = \ln\left(\frac{e^{0/8}+e^{-0/8}}{2}\right) = 0$$

this means that we have to satisfy the equation

$$f(0) = a - b\cdot 0 = 300$$

and thus $a = 300$. This means we only have to determine b using least squares! Had you gone crazy and immediately started using least squares for everything, you wouldn't have found $a = 300$ and your answer would simply have been wrong. Now, we have simply that the sum of the squares of the errors are

$$\psi(b) = \sum_{i=1}^{2}\left[h(t_i) - f(t_i)\right]^2 = \sum_{i=0}^{2}\left[h(t_i) - 300 + b\ln\left(\cosh\left(\frac{t_i}{8}\right)\right)\right]^2$$

Then, differentiating with respect to b:

$$\frac{\psi(b)}{b} = \sum_{i=1}^{2} 2\left[h(t_i) - 300 + b\ln\left(\cosh\left(\frac{t_i}{8}\right)\right)\right]\cdot\ln\left(\cosh\left(\frac{t_i}{8}\right)\right) = 0$$

Please note how remarkably easy the differentiation with respect to b is: the ln term is simply a constant when differentiating w.r.t. to b. Now, we have $h(t_1) = 220$ and $h(t_2) = 22$. Furthermore:

$$\begin{aligned}
\ln\left(\cosh\left(\frac{t_1}{8}\right)\right) &= \ln\left(\cosh\left(\frac{4}{8}\right)\right) = \ln\left(\frac{e^{4/8}+e^{-4/8}}{2}\right) = 0.12011\\
\ln\left(\cosh\left(\frac{t_2}{8}\right)\right) &= \ln\left(\cosh\left(\frac{8}{8}\right)\right) = \ln\left(\frac{e^{8/8}+e^{-8/8}}{2}\right) = 0.43378
\end{aligned}$$

We can rewrite the differential equation to

$$\begin{aligned}
\sum_{i=1}^{2} 2h(t_i)\cdot\ln\left(\cosh\left(\frac{t_i}{8}\right)\right) &= \sum_{i=1}^{2}\left[2\cdot 300\cdot\ln\left(\cosh\left(\frac{t_i}{8}\right)\right) + 2b\cdot\left(\ln\left(\cosh\left(\frac{t_i}{8}\right)\right)\right)^2\right]\\
2\cdot(220\cdot 0.12011 + 22\cdot 0.43378) &= 2\cdot 300\cdot 0.12011 - 2b\cdot 0.12011^2 + 2\cdot 300\cdot 0.43378 - 2b\cdot 0.43378^2\\
-0.405183b &= -260.39928
\end{aligned}$$

and thus $b = 642.67 \approx 643$ and thus H is correct.

1.8. Question 24

The correct answer is H. The linear regressor will have the form $f(x) = a + bx$, so that the sum of the least squares becomes:

$$\psi(a,b) = \sum_{i=1}^{4}\left[h(x_i) - f(x_i)\right]^2 = \sum_{i=1}^{4}\left[h(x_i) - a - bx_i\right]^2$$

Performing partial differentiation:

$$\frac{\partial \psi(a,b)}{\partial a} = -2\sum_{i=1}^{4}\left[h(x_i)-a-bx_i\right]=0$$
$$\frac{\partial \psi(a,b)}{\partial b} = -2\sum_{i=1}^{4}\left[h(x_i)-a-bx_i\right]x_i=0$$

The first equation can be rewritten to

$$\begin{aligned}\sum_{i=1}^{4} h(x_i) &= \sum_{i=1}^{4} a+\sum_{i=1}^{4} bx_i\\ 0.1+0.2-0.1+0.1 &= 4a+b\cdot 0+b\cdot 1+b\cdot 2+b\cdot 4\\ 4a+7b &= 0.3\end{aligned}$$

The second equation can be rewritten to

$$\begin{aligned}\sum_{i=1}^{4} h(x_i)x_i &= \sum_{i=1}^{4} ax_i+\sum_{i=1}^{4} bx_i^2\\ 0.1\cdot 0+0.2\cdot 1-0.1\cdot 2+4\cdot 0.1 &= 0a+1a+2a+4a+b\cdot 0^2+b\cdot 1^2+b\cdot 2^2+b\cdot 4^2\\ 7a+21b &= 0.4\end{aligned}$$

This forms the matrix equation

$$\begin{bmatrix}4 & 7 & 0.3\\ 7 & 21 & 0.4\end{bmatrix} \sim \begin{bmatrix}1 & 1.75 & 0.075\\ 7 & 21 & 0.4\end{bmatrix} \sim \begin{bmatrix}1 & 1.75 & 0.075\\ 0 & 8.75 & -0.125\end{bmatrix}$$

From the second row, we then quickly have $8.75b=-0.125$ and thus $b=-0.0142857$. From the first row:

$$a+1.75\cdot -0.0142857=0.075$$

and thus $a=0.1$, and thus

$$f(x)=0.1-0.0142857x$$

which has a root at $x=7$.

Part III

Quiz 2015

1.1. Question 1

The correct answer is D. The Taylor series you remember for $\sin(x)$,

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

is the Taylor series around $x = 0$. Thus, we have to derive it here again:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

with $x_0 = \pi/2$, we have

$$\begin{aligned} f(x) = \sin(x) \quad &\rightarrow \quad f(\pi/2) = \sin(\pi/2) = 1 \\ f'(x) = \cos(x) \quad &\rightarrow \quad f(\pi/2) = \cos(\pi/2) = 0 \\ f''(x) = -\sin(x) \quad &\rightarrow \quad f(\pi/2) = -\sin(\pi/2) = -1 \end{aligned}$$

and thus

$$f(x) = \frac{1}{0!}\left(x - \frac{\pi}{2}\right)^0 + \frac{0}{1!}\left(x - \frac{\pi}{2}\right)^1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

Integrating:

$$\begin{aligned} \int_0^\pi \sin(x)\,dx &= \int_0^\pi \left(1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2\right)dx = \left[x - \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3\right]_0^\pi = \left(\pi - \frac{1}{6}\left(\pi - \frac{\pi}{2}\right)^3\right) - \left(0 - \frac{1}{6}\left(0 - \frac{\pi}{2}\right)^3\right) \\ &= \left(\pi - \frac{1}{6}\frac{\pi^3}{8}\right) + \frac{1}{6}\left(-\frac{\pi^3}{8}\right) = \pi - \frac{\pi^3}{24} \end{aligned}$$

and thus answer D is correct. Note: you could have done this a bit more elegantly by realizing that $\int_0^\pi \sin(x)\,dx = \int_{-\pi/2}^{\pi/2} \cos(x)\,dx$ and making the Taylor expansion of $\cos(x)$ around $x = 0$ (which you already know by heart), which makes the integral slightly easier.

1.2. Question 2

The correct answer is C. Just remember that the Taylor series equals

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

but as we only include up to quadratic terms, we simply have

$$f(x) = x$$

Then just try each answer with your calculator:

$$\begin{aligned} \frac{\sin\left(\frac{7\pi}{180}\right) - \frac{7\pi}{180}}{\sin\left(\frac{7\pi}{180}\right)} \cdot 100\% &= -0.2492\% \\ \frac{\sin\left(\frac{14\pi}{180}\right) - \frac{14\pi}{180}}{\sin\left(\frac{14\pi}{180}\right)} \cdot 100\% &= -1.002\% \end{aligned}$$

and thus answer C is correct.

1.3. Question 3

The correct answer is B. We must have

$$\begin{aligned}
\left|\phi'(x)\right| &< 1 \\
\phi(x) &= x - \frac{\sin(x)}{\cos(x)} \\
\phi'(x) &= 1 - \frac{\cos^2(x) - -\sin^2(x)}{\cos^2(x)} = 1 - \frac{1}{\cos^2(x)} \\
\left|1 - \frac{1}{\cos^2(x)}\right| &< 1 \\
\frac{1}{\cos^2(x)} &< 2 \\
\cos^2(x) &> \frac{1}{2} \\
\cos(x) &> \frac{1}{2}\sqrt{2} \\
|x| &< \frac{\pi}{4}
\end{aligned}$$

and thus answer B is correct.

1.4. Question 4

The correct answer is D. The error after n iterations equals

$$\epsilon_n = e_0 \cdot \left(\frac{1}{4}\right)^n = 1 \cdot \left(\frac{1}{4}\right)^n$$

We must have

$$\begin{aligned}
1 \cdot \left(\frac{1}{4}\right)^N &< 2^{-32} \\
2^{-2N} &< 2^{32}
\end{aligned}$$

and thus at least 16 iterations are required and thus answer D is correct.

1.5. Question 5

The correct answer is D. For (a): all of them are perfectly valid (to be honest, I don't know exactly know when they'd be invalid; possible when you have $\phi = \sqrt{x+1}$ and then try $x_0 = 0.1$). However, only ϕ_2 will actually converge to 0: remember that we must have

$$\left|\phi'(x)\right| < 1$$

We have

$$\begin{aligned}
\phi_1' &= 1 + \cos(x) \\
\phi_2' &= 1 - \cos(x) \\
\phi_3' &= \frac{2x + \cos(x)}{2\sqrt{x^2 + \sin(x)}}
\end{aligned}$$

$\phi_1 \approx 2$ for small values of x; $\phi_2 \approx 0$ for small values of x, and ϕ_3 goes towards infinity for small values of x (just try on your calculator). Thus, answer D is correct.

1.6. Question 6

The correct answer is C. You may be inclined to automatically answer B, but that's wrong this time: remember we have

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

with $f(x) = x^{2m}$ and $f'(x) = 2mx^{2m-1}$, we get

$$\phi(x) = x - \frac{x^{2m}}{2mx^{2m-1}} = x - \frac{x}{2m}$$

and thus

$$\begin{aligned} \phi'(x) &= 1 - \frac{1}{2m} \\ \phi''(x) &= 0 \end{aligned}$$

Now plugging this in leads to

$$\begin{aligned} e_{N+1} &= \phi'(\tilde{x}) e_N + \frac{1}{2}\phi''(\tilde{x}) e_N^2 + O\left(e_N^3\right) \approx \left(1 - \frac{1}{2m}\right) e_N + \frac{1}{2} \cdot 0 \cdot e_N^2 = \left(1 - \frac{1}{2m}\right) e_N \\ e_{N+1} &= \frac{2m-1}{2m} e_N \end{aligned}$$

and thus the convergence is linear (as there is linearity between e_{N+1} and e_N) with rate $\frac{2m-1}{2m}$ and thus answer C is correct.

Now, you may be wondering, why is it not quadratic this time? Did I lie to you? No, I did not lie: x^{2m} is a special case: remember from the summary that we had

$$\phi'(x) = \frac{f(x) f''(x)}{(f'(x))^2}$$

and we *always* have $f(\tilde{x}) = 0$ (in this case, $f(0) = 0$. However, for x^{2m}, there's something special: $f'(0) = 2m \cdot 0^{2m-1} = 0$ as well. This clearly causes problems $\phi'(x)$ as you divide 0 by 0^2; this is what causes you to calculate the rate of convergence manually using the provided formula.

1.7. Question 7

The correct answer is C. We apparently have

$$\epsilon_2 \leq \max_{x \in [1,2]} \left| f^{(2+1)}(x) \right| \frac{\left| h^{2+1} \right|}{4(2+1)}$$

We have

$$\begin{aligned} f(x) &= x^{1/2} \\ f'(x) &= \frac{1}{2} \cdot x^{-1/2} \\ f''(x) &= -\frac{1}{4} \cdot x^{-3/2} \\ f'''(x) &= \frac{3}{8} \cdot x^{-5/2} = \frac{3}{8x^{5/2}} \end{aligned}$$

This has clearly a maximum at $x = 1$ (on the interval [1,2], $x = 1$ gives the largest value), so that

$$f'''(1) = \frac{3}{8 \cdot 1^{5/2}} = 0.375$$

We thus have

$$\begin{aligned} 0.375 \cdot \frac{h^3}{4 \cdot 3} = 0.03125 \cdot h^3 &\leq 5 \cdot 10^{-8} \\ h^3 &\leq 1.6 \cdot 10^{-6} \\ h &\leq 0.011696 \end{aligned}$$

and thus $h \approx 0.01$ and thus answer C is correct.

1.8. Question 8

The correct answer is E. $f(x)$ does not have to be linear nor quadratic, even though the fact that ϕ_1 is apparently a valid interpolating polynomial (which suggests that a linear line goes through the interpolation points). However, this may just be a case of choosing the nodes a certain way (for example, if you would sample $\sin(x)$ at $x = 0$, $x = \pi$, $x = 2\pi$, $x = 3\pi$, etc., then you would end up with an interpolating polynomial that would simply be zero, even though $\sin(x)$ is definitely not a zero function). In general, you can't say *anything* about the function $f(x)$ from interpolation. Meanwhile, if there's an interpolating polynomial, then it's always true that all interpolating polynomials are the same as long as the nodes or function does not change (so E is definitely correct).

1.9. Question 9

The correct answer is D. Although all methods of interpolation give the exact same results for the same function and nodes, regression absolutely gives you something different (as the regression formula aims to reduce the errors, but not actually go through the nodes). Different ways of regression also give you different results, so none of the functions will be the same.

1.10. Question 10

The correct answer is D. Let's use Newton's basis. We then have

$$p_2(x) = d_0 + d_1(x - x_0) + d_2(x - x_0)(x - x_1) = d_0 + d_1 x + d_2 x(x - 0.5)$$

and we have to solve the matrix equation

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_1 - x_0) \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} &= \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 1 & (0.5 - 0) & 0 \\ 1 & (1 - 0) & (1 - 0)(0.5 - 0) \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} &= \begin{bmatrix} \sin\left(\frac{1}{2}\pi \cdot 0\right) \\ \sin\left(\frac{1}{2}\pi \cdot 0.5\right) \\ \sin\left(\frac{1}{2}\pi \cdot 1\right) \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 1 & 0.5 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0.7071 \\ 1 \end{bmatrix}
\end{aligned}$$

From the first row, clearly $d_0 = 0$. From the second row, $d_1 = 1.414$. From the third row then:

$$1 \cdot 0 + 1 \cdot 1.414 + 0.5 \cdot d_2 = 1$$

leading to $d_2 = -0.8284$. We then have

$$p_2(x) = 0 + 1.414x - 0.8284x(x - 0.5) = -0.8284x^2 + 1.8284x$$

Setting this equal to zero:

$$\begin{aligned} -0.8284x^2 + 1.8284x &= 0 \\ x(-0.8284x + 1.8284) &= 0 \end{aligned}$$

which has solutions $x = 0$ (the trivial solution) and $x = 2.207$. Thus, answer D is correct.

Part IV

Quiz 2016

1.1. Question 1

The correct answer is G. The Taylor series for a sine function around $x_0 = 0$ is

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Note that this immediately excludes answers B, D, F and H: we can only truncate after an odd power of $(x - x_0)^i$. We have $\sin(\pi/4) = 0.7071$. Now, let's just go one-by-one what the error is for each value of k (1, 3 or 5). For $k = 1$, we'd get $T_1[f,0](\pi/4) = \pi/4 \approx 0.7854$, which is clearly more than 10^{-3} larger than 0.7071. We have for $k = 3$ that $T_3[f,0](\pi/4) = \pi/4 - (\pi/4)^3/3! \approx 0.7047$, which is still more than 10^{-3} off of the correct answer. Thus, answer G is correct (you can verify yourself that the error is then small enough).

1.2. Question 2

The correct answer is E. First of all, you'd be a complete idiot if you'd say C, but let's put that aside. Note that the program actually still functions: it'll just always follow the "else" line, meaning that each time, a takes the value of c. Now, please note, this is *pseudo-code*, not Python-code. That means that the for-loop, $i = [0:N]$ means that i runs from 0 till and including N (whereas in Python-code, it means that you have to stop at N (thus not include N itself). So, for a certain N, the program computes c_i $N+1$ times. What happens in the computation of c_i? We have that

$$c_{i+1} = \frac{b_{i+1} + a_{i+1}}{2}$$

but we'll always have $a_{i+1} = c_i$ and $b_{i+1} = b_i$ (thus $b_{i+1} = b$). This means that for example, we get the following sequence:

$$\begin{aligned}
c_0 &= \frac{b+a}{2} = a_1 \\
c_1 &= \frac{b+a_1}{2} = \frac{b + \frac{b+a}{2}}{2} = a_2 \\
c_2 &= \frac{b+a_2}{2} = \frac{b + \frac{b+\frac{b+a}{2}}{2}}{2} = \frac{b}{2} + \frac{\frac{b+\frac{b+a}{2}}{2}}{2} = \frac{b}{2} + \frac{b}{4} + \frac{b+a}{8} = b + \frac{a-b}{2^3}
\end{aligned}$$

From this, it's clear that the general formula is

$$b - \frac{b-a}{2^{N+1}}$$

Alternatively, the more straightforward approach is to realize that c_i is simply b minus the length of the interval $b-a$ divided by the number of times the interval has been cut in half, i.e. $N+1$ times.

1.3. Question 3

The correct answer is F. We know that the real root is located at $x = \sqrt[3]{9} \approx 2.08008$. From Newton's method:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We have $f'(x) = 3x^2$, so that

$$x_1 = x_0 - \frac{x_0^3 - 9}{3x_0^2} = 2 - \frac{2^3 - 9}{3 \cdot 2^2} = 2.0833$$

and thus the error is approximately -0.00325, and thus answer F is correct.

1.4. Question 4

The correct answer is C. Remember question 6 of the 2015 quiz. The error was related via

$$\epsilon_{N+1} = \phi'(\tilde{x}) e_N + \frac{1}{2}\phi''(\tilde{x}) e_N^2 + O\left(e_N^3\right)$$

where

$$\phi'(x) = \frac{f(x) f''(x)}{\left(f'(x)\right)^2}$$

Now, normally, on the condition that $f'(x) \neq 0$, the rate of convergence is quadratic, this means that $\phi'(x) \to 0$, as $f(\tilde{x}) = 0$, thus meaning that in the relation for the error, we only have the quadratic term left. However, we saw that for monomials, this was not the case, because there we always had $f'(0) = 0$, and therefore $\phi'(x) \neq 0$ (because we divide 0 by 0^2 and we don't know who wins). So, let's use the hint the quiz gives to you: calculate the derivative of $f(x) = \tan x - x$:

$$\begin{aligned}
f(x) &= \frac{\sin(x)}{\cos(x)} - x \\
f'(x) &= \frac{\cos^2(x) - \sin(x) \cdot - \sin(x)}{\cos^2(x)} - 1 = \frac{1}{\cos^2(x)} - 1 \\
f'(0) &= \frac{1}{\cos^2(0)} - 1 = 0 \\
f'\left(\frac{3\pi}{2}\right) &= \frac{1}{\cos^2\left(\frac{3\pi}{2}\right)} - 1
\end{aligned}$$

and the second one is discontinuous at $3\pi/2$ so your calculator gives an error. What does this tell us? For $\hat{x} = 0$, $f'(x) = 0$ and therefore $\phi'(x) \neq 0$. For $\hat{x} = 3/\pi/2$, this means that the convergence is quadratic, as we know that $f'(x) \neq 0$, and thus we do have $\phi'(x) = 0$, and thus we only have quadratic terms in our expression for the convergence.

Now, you may wonder, but for the first one, if we have

$$e_{N+1} = \phi'(\tilde{x}) e_N + \frac{1}{2}\phi''(\tilde{x}) e_N^2 + O\left(e_N^3\right)$$

how do we know for certain that $\phi''(\tilde{x})$ is equal to zero? Well, we don't, but the dominant term in the equation is then the lowest-order term (e_N will be small, so e_N^2 will be really small, meaning that e_{N+1} mainly depends on e_N and not e_N^2).

1.5. Question 5

The correct answer is C. As we are allowed to use our graphical calculators after all (I didn't know that before, so that's why I solved everything manually), let's just use the monomial basis, as that's the easiest one to construct the matrix for (of course, you're also allowed to use Newton's basis, or Lagrange if you're feeling like a bad-ass).

We assume the form $p(x) = a_0 + a_1 x + a_2 x^2$, meaning we get the system of equations

$$\begin{aligned}
p(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 = a_0 &= f(0) = \cos(\pi \cdot 0) = 1 \\
p\left(\frac{1}{2}\right) = a_0 + a_1 \cdot \frac{1}{2} + a_2 \cdot \left(\frac{1}{2}\right)^2 = a_0 + \frac{a_1}{2} + \frac{a_2}{4} &= f\left(\frac{1}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0 \\
p(1) = a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 = a_0 + a_1 + a_2 &= f(1) = \cos(\pi) = -1
\end{aligned}$$

In other words::

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 1 & \frac{1}{2} & \frac{1}{4} & | & 0 \\ 1 & 1 & 1 & | & -1 \end{bmatrix}$$

Solving with our graphical calculator yields $a_0 = 1$, $a_1 = -2$ and $a_2 = 0$. Thus,

$$p(x) = 1 - 2x$$

and we thus get

$$\epsilon\left(\frac{1}{4}\right) = \left|f\left(\frac{1}{4}\right) - p\left(\frac{1}{4}\right)\right| = \left|\cos\left(\frac{\pi}{4}\right) - \left(1 - 2\cdot\frac{1}{4}\right)\right| = \frac{\sqrt{2}}{2} - \frac{1}{2}$$

and thus answer C is correct.

1.6. Question 6

The correct answer is F. Take the polynomial $f(x) = x^2$. We then get the system of equations

$$\begin{aligned} p(0) = a_0 \cdot 1 + a_0 \cdot \cos(0) + a_2 \cdot \sin(0) = a_0 + a_1 &= 0^2 = 0 \\ p\left(\frac{\pi}{2}\right) = a_0 \cdot 1 + a_1 \cdot \cos\left(\frac{\pi}{2}\right) + a_2 \cdot \sin\left(\frac{\pi}{2}\right) = a_0 + a_2 &= \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4} \\ p(\pi) = a_0 \cdot 1 + a_1 \cdot \cos(\pi) + a_2 \cdot \sin(\pi) a_0 - a_1 &= \pi^2 \end{aligned}$$

so that we have the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 1 & 0 & 1 & | & \frac{\pi^2}{4} \\ 1 & -1 & 0 & | & \pi^2 \end{bmatrix}$$

Solving with our graphical calculator yields $a_0 = \pi^2/2$, $a_1 = -\pi^2/2$ and $a_2 = -\pi^2/4$, so that

$$p(x) = \frac{\pi^2}{2} - \frac{\pi^2}{2}\cos(x) - \frac{\pi^2}{4}\sin(x)$$

Differentiating and plugging in $x = 0$ gives:

$$\begin{aligned} p'(x) &= \frac{\pi^2}{2}\sin(x) - \frac{\pi^2}{4}\cos(x) \\ p'(0) &= \frac{\pi^2}{2}\sin(0) - \frac{\pi^2}{4}\cos(0) = -\frac{\pi^2}{4} \end{aligned}$$

All in all, the correct answer is F.

1.7. Question 7

The correct answer is D. Apparently (something which you could also observe in questions 5 and 6 btw), the matrix looks like

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \phi_2(x_0) & \phi_3(x_0) & \cdots \\ \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \phi_3(x_1) & \cdots \\ \phi_0(x_2) & \phi_1(x_2) & \phi_2(x_2) & \phi_3(x_2) & \cdots \\ \phi_0(x_3) & \phi_1(x_3) & \phi_2(x_3) & \phi_3(x_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \phi_0(x_N) & \phi_1(x_N) & \phi_2(x_N) & \phi_3(x_N) & \cdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_N) \end{bmatrix}$$

But, apparently, it's a lower triangular matrix, i.e. it looks like

$$\begin{bmatrix} \phi_0(x_0) & 0 & 0 & 0 & \cdots \\ \phi_0(x_1) & \phi_1(x_1) & 0 & 0 & \cdots \\ \phi_0(x_2) & \phi_1(x_2) & \phi_2(x_2) & 0 & \cdots \\ \phi_0(x_3) & \phi_1(x_3) & \phi_2(x_3) & \phi_3(x_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \phi_0(x_N) & \phi_1(x_N) & \phi_2(x_N) & \phi_3(x_N) & \cdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_N) \end{bmatrix}$$

This means that $\phi_3(x_0) = 0$, $\phi_3(x_1) = 0$ and $\phi(x_2) = 0$, meaning $\phi_3(x)$ has roots at x_0, x_1 and x_2. Answer D is thus correct.

1.8. Question 8

The correct answer is C. We have that $p(3) = 20$, in other words:

$$p(3) = a_0 + a_1(3-3) + 2\cdot(3-3)(3-5) = a_0 = 20$$

so that $a_0 = 20$. We then have for $x = 5$:

$$p(5) = 20 + a_1(5-3) + 2\cdot(5-3)(5-5) = 20 + 2a_1 = 12$$

and thus $a_1 = -4$. We then have

$$\begin{aligned} p(x_2) = 20 - 4(x_2-3) + 2\cdot(x_2-3)(x_2-5) &= 20 - 4x_2 + 12 + 2x_2^2 - 16x_2 + 30 = 2x^2 - 20x_2 + 62 = 30 \\ x^2 - 10x_2 + 16 &= 0 \\ (x-5)^2 - 9 &= 0 \\ x - 5 &= \pm 3 \\ x_2 &= 5 \pm 3 \end{aligned}$$

and thus x_2 can be either 2 or 8, and thus answer C is correct.

1.9. Question 9

The correct answer is F. This question really comes down to your reading comprehension skills: although Weierstrass theorem states that you can approximate *any* function $f(x)$ that is continuous (but not necessarily differentiable) on an interval $[a,b]$ with a polynomial where the error tends to zero. However, it does not tell you *how* to do it: if you use a uniform grid, it does not always tend to zero: only if you choose your grid in a specific way does Weierstrass theorem hold. Therefore, answer F is correct.

1.10. Question 10

The correct answer is B. You may wonder, how can we know ξ? We can find its value from the fact that we actually know the error at $x = \frac{1}{4}$. First, we need to find the interpolant however. Again, let's just the monomial basis, i.e. $p(x) = a_0 + a_1x + a_2x^2$. We then get the set of equations

$$\begin{aligned} p(0) = a_0 + a_1\cdot 0 + a_2\cdot 0^2 = a_0 &= f(0) = \sin(\pi\cdot 0) = 0 \\ p\left(\frac{1}{2}\right) = a_0 + a_1\cdot\frac{1}{2} + a_2\cdot\left(\frac{1}{2}\right)^2 = a_0 + \frac{a_1}{2} + \frac{a_2}{4} &= f\left(\frac{1}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \\ p(1) = a_0 + a_1\cdot 1 + a_2\cdot 1^2 = a_0 + a_1 + a_2 &= f(1) = \sin(\pi\cdot 1) = 0 \end{aligned}$$

so that we get

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & | & 1 \\ 1 & 1 & 1 & | & 0 \end{bmatrix}$$

for which the solution is $a_0 = 0$, $a_1 = 4$ and $a_2 = -4$, and thus

$$\begin{aligned} p(x) &= 4x - 4x^2 \\ p\left(\frac{1}{4}\right) &= 4 \cdot \frac{1}{4} - 4 \cdot \left(\frac{1}{4}\right)^2 = \frac{3}{4} \end{aligned}$$

so that

$$f\left(\frac{1}{4}\right) - p_2\left(\frac{1}{4}\right) = \frac{\sqrt{2}}{2} - \frac{3}{4} \approx -0.042893$$

Furthermore, we have:

$$\begin{aligned} f^{(2+1)}(\xi) &= -\pi^3 \cos(\pi\xi) \\ (2+1)! &= 6 \\ \prod_{i=0}^{N}\left(\frac{1}{4} - x_i\right) &= \left(\frac{1}{4} - 0\right)\left(\frac{1}{4} - \frac{1}{2}\right)\left(\frac{1}{4} - 1\right) = 0.046875 \end{aligned}$$

Thus, we must solve

$$\begin{aligned} -0.042893 &= \frac{-\pi^3 \cos(\pi\xi)}{6} \cdot 0.046875 \\ \cos(\pi\xi) &= 0.177 \end{aligned}$$

Of course, you can now find ξ, then plug this into $\cos(\pi\xi)$ but that is rather pointless. Anyway, answer B is correct.

Note: you could have find $p_2(x)$ in a different way actually. We know that $f(0) = 0$ and $f(1) = 0$: this means that the interpolating function *must* be of the form

$$p(x) = a(x-0)(x-1) = ax^2 - ax$$

We use then the condition that $p(1/2) = 1$ to find that

$$\begin{aligned} p\left(\frac{1}{2}\right) = a \cdot \left(\frac{1}{2}\right)^2 - a \cdot \frac{1}{2} &= 1 \\ -\frac{1}{4}a = 1 \end{aligned}$$

so that $a = -4$: the interpolating function is thus

$$p(x) = -4x^2 + 4x$$

This is quicker than regular methods, but its use is much more circumstantial than the methods you already know.