Part I

Module 5

1.1. Question 1

The correct answer is E: clearly, as we have y''', it's 3rd order. Furthermore, we don't have absolutely crazy stuff like $\sin(y'')$, $\ln(y'')$ in it, but rather just simple linear relations, thus it's linear. Finally, as we have $y'(1) = 0$, it's no longer an initial value problem, but a boundary value problem! Thus, answer E is correct.

1.2. Question 2

The correct answer is E. Neither $f(y) = |1/y|$, $f(y) = \tan y$ nor $f(y) = H(y)$ are continuous themselves, so that only leaves the first three. $f(y) = |y|^{1/3}$ and $f(y) = y^{1/3}$ are *both* continuous, but their derivative grows unbounded near $y = 0$: thus, the there is no L large enough such that

$$\left|f(y_1) - f(y_2)\right| \leq L\left|y_1 - y_2\right|$$

since it would require L to be infinity (which does not make sense) if both y_1 and y_2 are very close to 0.

Note that $f(y) = |y|$ is continuous, and also Lipschitz continuous: its derivative never grows unbounded, so there is always a L such that the above condition is met. However, do note that it is also now apparent why Lipschitz continuity is a weaker property than differentiability: $f(y) = |y|$ is *not* differentiable (at $y = 0$ the derivative is not properly defined).

1.3. Question 3

The correct answer is C. The Cauchy-Lipschitz theorem states that there is unique solution to an IVP on an interval if $f(y)$ is Lipschitz continuous on the interval.

1.4. Question 4

The correct answer is G: we have that

$$c = f_y = \frac{\partial f(x,y)}{\partial y} = \frac{\partial(2\lambda y)}{\partial y} = 2\lambda < 0$$

which is satisfied for $\lambda < 0$. Thus, answer G is correct.

1.5. Question 5

The correct answer is C: we have to linearise e^y around y_n. We have $f(y) = e^y$, and thus applying a Taylor expansion, going up to $N = 1$ as we are only interest in linear terms, we get

$$f(y) = \frac{e^{y_n}}{0!}(y - y_n)^0 + \frac{e^{y_n}}{1!}(y - y_n)^1 + O(y - y_n)^2 = e^{y_n}(y - y_n + 1)$$

Substituting leads to

$$\begin{aligned} y' + 2y &= -e^{y_n}(y - y_n + 1) \\ y' + (2 + e^{y_n})y &= e^{y_n}(y_n - 1) \end{aligned}$$

and thus answer C is correct.

1.6. Question 6

The correct answer is B. A is wrong because using new (unknown) values is exactly what makes a method implicit. C is wrong because multistep methods need more than initial conditions. D is clearly wrong. E is just false, F is false, and G is wrong as well.

1.7. Question 7

The correct answer is B: the global error will be of one order less. The correct answer is thus B, because you'd expect it to be one order lower.

1.8. Question 8

The correct answer is D: we have the series expansions

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + O\left(h^4\right) \\ y_{n-1} &= y_n - hy'_n + \frac{h^2}{2}y''_n - \frac{h^3}{6}y'''_n + O\left(h^4\right) \\ f(y_n) &= y'_n \end{aligned}$$

Thus, plugging this into the scheme:

$$\begin{aligned} y_{n+1} &= y_{n-1} + f(y_n) \\ y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + O\left(h^4\right) &= y_n - hy'_n + \frac{h^2}{2}y''_n - \frac{h^3}{6}y'''_n + O\left(h^4\right) + 2hy'_n \\ \frac{h^3}{6}y'''_n + O\left(h^4\right) &= -\frac{h^3}{6}y'''_n + O\left(h^4\right) \end{aligned}$$

This means that we have to add a term of order $O\left(h^3\right)$ to have the left-side of the equation equal to the right side. Therefore, the order of the local truncation error is $O\left(h^3\right)$ and answer D is correct.

1.9. Question 9

The correct answer is B: the Taylor expansions are

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + O\left(h^4\right) \\ y_n &= y_n \\ y_{n-1} &= y_n - hy'_n + \frac{h^2}{2}y''_n - \frac{h^3}{6}y'''_n + O\left(h^4\right) \\ y_{n-2} &= y_n - 2hy'_n + 2h^2y''_n - \frac{4h^3}{3}y'''_n + O\left(h^4\right) \\ f(y_n) &= y'_n \end{aligned}$$

Then, plugging this into the scheme:

$$\begin{aligned} y_{n+1} &= \frac{6y_n - y_{n-1} - y_{n-2} + hf(y_n)}{4} \\ y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + O\left(h^4\right) &= \frac{6y_n - y_n + hy_n' - \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + O\left(h^4\right)}{4} \\ &= + \frac{-y_n + 2hy_n' - 2h^2y_n'' + \frac{4h^3}{3}y_n''' + O\left(h^4\right) + hy_n'}{4} \\ \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + O\left(h^4\right) &= -\frac{5h^2}{8}y_n'' + \frac{3h^3}{8}y_n''' + O\left(h^4\right) \end{aligned}$$

The truncation error is what you have to add to the right-side of the equation to make the two sides of the equations equal; clearly, the first term in it will be

$$\frac{h^2}{2}y_n'' - \left(-\frac{5h^2}{8}y_n''\right) = \frac{9}{8}y''(x_n)h^2$$

as then the terms involving y_n'' will cancel out. Thus, answer B is correct.

1.10. Question 10

The correct answer is C: remember that Euler's forward scheme was

$$y_{n+1} = y_n + \Delta t \cdot f(y_n)$$

We have $f(y_n) = -3y + 1$, and thus

$$y_{n+1} = y_n + 0.001(-3y + 1) = y_n - 0.003y_n + 0.001$$

and thus answer C is correct.

1.11. Question 11

The correct answer is C: remember that Euler's forward scheme was

$$y_{n+1} = y_n + \Delta t \cdot f(y_n)$$

We have $f(y_n) = 2y - 5$, and thus

$$y_{n+1} = y_n + 0.002(2y - 5) = y_n + 0.004y_n - 0.01 = 1.004y_n - 0.01$$

and thus answer C is correct.

1.12. Question 12

Answer C is correct: note that we use the scheme

$$y_{n+1} = y_n + \Delta t f(y_{n+1}) = y_n + 0.01 \cdot -0.1y_{n+1}$$

We can find an explicit solution for this:

$$\begin{aligned} y_{n+1} + 0.001y_{n+1} &= y_n \\ y_{n+1} &= \frac{y_n}{1.001} \end{aligned}$$

Thus, answer C is correct.

1.13. Question 13

The correct answer is D: we simply have $y_0 = 1$. Then, $f(y_0) = -100y_0 = -100 \cdot 1 = -100$, and thus

$$y_1 = y_0 + \Delta t \cdot f(y_0) = 1 + 0.001 \cdot -100 = 0.9$$

Then, $f(y_1) = -100y_1 = -100 \cdot 0.9 = -90$, and thus

$$y_2 = y_1 + \Delta t \cdot f(y_1) = 0.9 + 0.001 \cdot -90 = 0.81$$

Finally, $f(y_2) = -100y_2 = -100 \cdot 0.81 = -81$, and thus

$$y_3 = y_2 + \Delta t \cdot f(y_2) = 0.81 + 0.001 \cdot -81 = 0.729$$

and thus answer D is correct.

1.14. Question 14

The correct answer is D: note that we use the scheme

$$y_{n+1} = y_n + \Delta t f(y_{n+1}) = y_n + 0.001 \cdot -100y_{n+1}$$

We can rewrite this to an explicit expression for y_{n+1}:

$$\begin{aligned} y_{n+1} + 0.1y_{n+1} &= y_n \\ y_{n+1} &= \frac{y_n}{1.1} \end{aligned}$$

Thus, with $y_0 = 1$, this leads to

$$\begin{aligned} y_1 &= \frac{y_0}{1.1} = \frac{1}{1.1} = 0.90909 \\ y_2 &= \frac{y_1}{1.1} = \frac{0.90909}{1.1} = 0.82645 \\ y_3 &= \frac{y_2}{1.1} = \frac{0.82645}{1.1} = 0.7513 \end{aligned}$$

and thus answer D is correct.

1.15. Question 15

The correct answer is C: this question may seem intimidating because you are seeing *both* a matrix equation *and* complex numbers, but it's actually easier than it seems. What you need to remember is that our solutions will look like

$$\begin{aligned} y_1(t) &= c_{1,1}e^{\lambda_1 t} + c_{1,2}e^{\lambda_2 t} = y_{1,1}(t) + y_{1,2}(t) \\ y_2(t) &= c_{2,1}e^{\lambda_1 t} + c_{2,2}e^{\lambda_2 t} = y_{2,1}(t) + y_{2,2}(t) \end{aligned}$$

Now, f_y is rather hard to define exactly; the result should be a scalar, and it's relatively difficult to come up with a scalar that represents this derivative. However, if we differentiate with respect to t, we get

$$\begin{aligned} y_1'(t) &= c_{1,1}\lambda_1 e^{\lambda_1 t} + c_{1,2}\lambda_2 e^{\lambda_2 t} \\ y_2'(t) &= c_{2,1}\lambda_1 e^{\lambda_1 t} + c_{2,2}\lambda_2 e^{\lambda_2 t} \end{aligned}$$

Now, you have to carefully compare this with the two solutions I wrote down just before. If you look well, we can write

$$\begin{aligned} y_1'(t) &= \lambda_1 y_{1,1}(t) + \lambda_2 y_{1,2}(t) \\ y_2'(t) &= \lambda_1 y_{1,1}(t) + \lambda_2 y_{1,2}(t) \end{aligned}$$

Compare this with $y' = f_y y$. It makes sense to decompose this problem into two separate parts: once, we use $y_1'(t) = \lambda_1 y_{1,1}(t)$ to deduce $f_y = \lambda_1$; the second time we use $y_1'(t) = \lambda_2 y_{1,2}(t)$ to deduce $f_y = \lambda_2$; we then check for what values of a the stability region is satisfied. Why is this decomposition allowed? As we simply sum the two parts $y_{1,1}$ and $y_{1,2}$ (we take a linear combination to be precise), $y_1'(t)$ will not be unstable if neither of the two parts are unstable.

So, let's first consider $f_y = a(1+i)$. We must then simply have

$$\begin{aligned} \frac{1}{2}|z+2| &< 1 \\ |0.01a(1+i)+2| &< 2 \end{aligned}$$

Now, taking the absolute value of a number x is taking $\sqrt{\mathrm{Re}(x)+\mathrm{Im}(x)}$; in this case

$$\begin{aligned} \sqrt{(0.01a+2)^2+(0.01a)^2} &< 2 \\ \sqrt{4+0.04a+0.0002a^2} &< 2 \\ 4+0.04a+0.0002a^2 &< 4 \\ a^2+200a &< 0 \\ a(a+200) &< 0 \end{aligned}$$

This has solutions $a = 0$ and $a = -200$, and is smaller than 0 for $0 < a < -200$, leading to answer C being correct. You can verify your answer by checking $f_y = a(1-i)$; however, as they are complex conjugates, it'll produce exactly the same result, which you'll quickly see during calculations. Now, if you want a more exotic way of looking at things, consider the following. $\frac{1}{2}|z+2| < 1$ is a circle of radius 2, centred at $z = -2$ in the complex plane. We can draw the lines $x = 0.01a(1+i)$ and $x = 0.01a(1-i)$ (the 0.01 coming from h) as shown in figure 1.1. Clearly, if we focus on the points on the line associated with $x = 0.01a(1-i)$, we see that the line lays inside the circle when the real part of x has values between $(-2,0)$ (the same holds for the line $x = 0.01a(1+i)$. This means that $-200 < a < 0$. One could also look at the imaginary part, and again one will quickly realize that only the points with $-200 < a < 0$ lay inside the circle. Therefore, answer C is correct.

1.16. Question 16

The correct answer is A: we first rewrite to

$$y_{n+1} - y_n = \Delta y_n = \frac{\Delta t}{2}\left[f(y_{n+1}) + f(y_n)\right]$$

Furthermore, Taylor expansions of the remaining term on the right-hand side is simply (linearising, thus truncating already after the linear term)

$$f(y_{n+1}) = f(y_n) + \Delta y_n f_y(y_n) + \mathcal{O}\left(\Delta y^2\right)$$

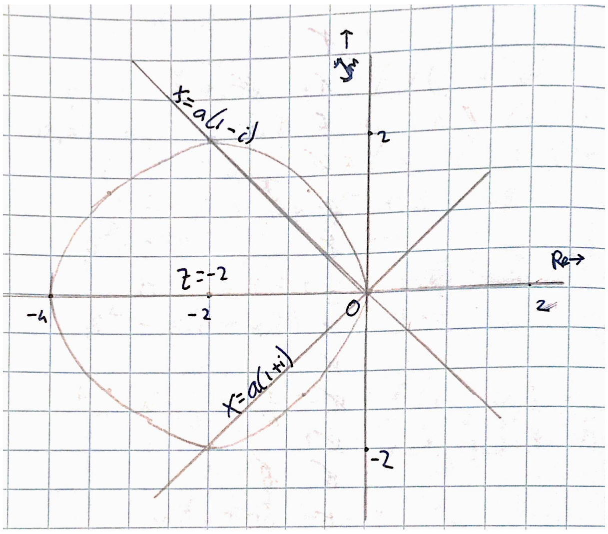

Figure 1.1: Stability region. No complaints about my circle pls.

Upon combination, this becomes

$$\Delta y_n = \frac{\Delta t}{2}\left[f(y_n)+\Delta y_n f_y(y_n)+O\left(\Delta y^2\right)+f(y_n)\right]$$
$$\Delta y_n = \frac{\Delta t}{2}\left[2f(y_n)+\Delta y_n f_y(y_n)+O\left(\Delta y^2\right)\right]$$
$$\left[1-\frac{\Delta}{2}f_y(y_n)\right]\Delta y_n = \Delta t f(y_n)+\Delta t O\left(\Delta y^2\right)$$

So answer A is correct. Now, you may wonder, what's up with the error? Shouldn't that be $O\left(\Delta t^3\right)$? Well, Δt is of the same order of magnitude as Δy (since we linearise, if we decrease Δt by a factor $\frac{1}{2}$, we also expect Δy to be decreased by a factor $\frac{1}{2}$, thus it's of the same order), and thus we actually do have $\Delta t O\left(\Delta y^2\right) = O\left(\Delta t^3\right)$.

1.17. Question 18

The correct answer is A: we first focus on rewriting the top equation. For that one, we use $z_1 = h$ and $z_2 = \dot{h}$ ($\dot{h}$ is simply h'). Then, according to what I've explained just before, the system for these two equations become

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} z_2 = \dot{h} \\ z_3 = \ddot{h} \end{bmatrix}$$

Now, from the ODE itself, we have

$$m\ddot{h}+K_h h = -L$$
$$\ddot{h} = \frac{-L}{m}-\frac{K_h}{m}h$$

Thus, we essentially get the system of equations

$$z_1' = \dot{h}$$
$$z_2' = \ddot{h} = -\frac{K_h}{m}h-\frac{L}{m}$$

or shortly

$$\begin{aligned} z_1' &= \dot{h} \\ z_2' &= -\frac{K_h}{m}h - \frac{L}{m} \end{aligned}$$

Or, in matrix form,

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{K_h}{m} & 0 \end{bmatrix} \begin{bmatrix} h \\ \dot{h} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{L}{m} \end{bmatrix}$$

The same process can be applied for the second equation; we use $z_4 = \theta$ and $z_5 = \dot{\theta}$. Then, we get

$$\begin{bmatrix} z_4' \\ z_5' \end{bmatrix} = \begin{bmatrix} z_5 = \dot{\theta} \\ z_6 = \ddot{\theta} \end{bmatrix}$$

Again, we have

$$\begin{aligned} I_\theta \ddot{\theta} + K_\theta \theta &= M \\ \ddot{\theta} &= \frac{M}{I_\theta} - \frac{K_\theta}{I_\theta}\theta \end{aligned}$$

and thus

$$\begin{aligned} z_4' &= \dot{\theta} \\ z_5' &= -\frac{K_\theta}{I_\theta}\theta + \frac{M}{I_\theta} \end{aligned}$$

Adding this to the previous matrix, we can write the matrix equation as

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_h}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{K_\theta}{I_\theta} & 0 \end{bmatrix} \begin{bmatrix} h \\ \dot{h} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{L}{m} \\ 0 \\ \frac{M}{I_\theta} \end{bmatrix}$$

Thus, clearly, answer A is correct.

1.18. Question 19

The correct answer is G. What you have to realize is that this is a linear system of equations, and you need to bear in mind that the solution for $x_1(t)$, $x_2(t)$, etc. will be of the form

$$x_j(t) = \sum_{i=1}^{4} c_{i,j} e^{\lambda_i t}$$

and thus we require λ_i for $i = 1, \dots, 4$ to be negative in order to be stable. Now, λ may be complex if you remember correctly; however, the imaginary part of this complex number would go to a term related with cos and sin (which always remains stable), whereas the real part of the complex number would be involved in the exponent of e; thus, specifically, we require $\text{Re}(\lambda) < 0$ for all eigenvalues of A. Thus, answer G is correct.

Part II

Module 6

1.19. Question 1

The correct answer is D: for 1D, Newton's method reduces to

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)}$$

We have

$$\begin{aligned} f(x) &= x^3 e^{-x} - \sin(x) \\ f'(x) &= 3x^2 e^{-x} - x^3 e^{-x} - \cos(x) \\ f''(x) &= 6xe^{-x} - 3x^2 e^{-x} - 3x^2 e^{-x} + x^3 e^{-x} + \sin(x) = e^{-x}\left(x^3 - 6x^2 + 6x\right) + \sin(x) \end{aligned}$$

so that

$$\begin{aligned} f'(5.5) &= 3 \cdot 5.5^2 \cdot e^{-5.5} - 5.5^3 \cdot e^{-5.5} - \cos(5.5) = -1.0177 \\ f''(5.5) &= e^{-5.5} \cdot \left(5.5^3 - 6 \cdot 5.5^2 + 6 \cdot 5.5\right) + \sin(5.5) = -0.6325 \end{aligned}$$

and thus

$$x_1 = 5.5 - \frac{-1.0177}{-0.6325} = 3.891$$

and thus answer D is correct.

1.20. Question 2

The correct answer is C: just literally look up the definition of the Hessian matrix.

1.21. Question 3

The correct answer is A: remember that the gradient is defined as

$$\nabla f(\bar{x}_n) = \begin{bmatrix} \frac{\partial f(\bar{x}_n)}{\partial u} \\ \frac{\partial f(\bar{x}_n)}{\partial v} \\ \frac{\partial f(\bar{x}_n)}{\partial w} \end{bmatrix}$$

Thus, let's evaluate each of those. First, write

$$f(u,v,w) = \left(u^2 + v\sin(w) + uvw\right)^{\frac{1}{2}}$$

as this is easier to apply the chain rule. First, for the derivative with respect to u:

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{1}{2}\left(u^2 + v\sin(w) + uvw\right)^{-\frac{1}{2}} \cdot (2u + vw) \\ \frac{\partial f(14,5,3)}{\partial u} &= \frac{1}{2}\left(14^2 + 5 \cdot \sin(3) + 14 \cdot 5 \cdot 3\right)^{-\frac{1}{2}} \cdot (2 \cdot 14 + 5 \cdot 3) = 1.0661 \end{aligned}$$

Similarly, for the derivative with respect to v:

$$\begin{aligned} \frac{\partial f}{\partial v} &= \frac{1}{2}\left(u^2 + v\sin(w) + uvw\right)^{-\frac{1}{2}} \cdot (\sin(w) + uw) \\ \frac{\partial f(14,5,3)}{\partial v} &= \frac{1}{2}\left(14^2 + 5 \cdot \sin(3) + 14 \cdot 5 \cdot 3\right)^{-\frac{1}{2}} \cdot (\sin(3) + 14 \cdot 3) = 1.0448 \end{aligned}$$

and thus answer A is correct. If you wanna continue anyway, we have for the derivative with respect to w:

$$\begin{aligned}\frac{\partial f}{\partial w} &= \frac{1}{2}\left(u^2+v\sin(w)+uvw\right)^{-\frac{1}{2}}\cdot(v\cos(w)+uv)\\ \frac{\partial f(14,5,3)}{\partial w} &= \frac{1}{2}\left(14^2+5\cdot\sin(3)+14\cdot5\cdot3\right)^{-\frac{1}{2}}\cdot(5\cdot\cos(3)+14\cdot5)=1.6128\end{aligned}$$

and thus confirmation again that A is correct.

1.22. Question 4

The correct answer is B: note that $f(x_{L_0})<f(x_{R_0})$; thus, the minimum will be between a_0 and $f(x_{R_0})$. This means that $a_1=-2$ and $b_1=-0.1459$. Furthermore, as we use the golden ratio, we have $x_{R_1}=x_{L_0}=-0.8541$. Finally, x_{L_1} will be at an equally large distance from a_1 as x_{R_1} is from b_1; the latter distance equals $-0.1459--0.841=0.7082$, and thus $x_{L_1}=-2+0.7082=-1.2918$. Thus, answer B is correct.

1.23. Question 5

The correct answer is D: no need to do the iterations yourself; remember that the interval is reduced by a factor

$$\phi^{-1}=\left(\frac{1+\sqrt{5}}{2}\right)^2=0.6180$$

and thus if the initial interval length is $10--10=20$, we get after three iterations

$$20\cdot0.6180^3=4.7214$$

and thus answer D is correct.

1.24. Question 6

The correct answer is A: no need to do the iterations yourself; remember that the interval is reduced by a factor

$$\phi^{-1}=\left(\frac{1+\sqrt{5}}{2}\right)^2=0.6180$$

and, after 10 iterations, the upper limit is

$$22+(113-22)\cdot0.6180^3=22.7399$$

and thus answer A is correct.

1.25. Question 7

The correct answer is G: this is a basic property of the Golden-search method. The initial error is at most $(b-a)/2$, since any point within the interval is at most $|(b-a)/2|$ away from the midpoint of the interval. Then, each iteration, the width of the interval decreases by a factor ϕ, so you have to divide by ϕ^n.

1.26. Question 8

The correct answer is B: this lies at the foundation of the steepest descent method.

1.27. Question 9

The correct answer is D: first of all, we have

$$f'(\mathbf{x}_0) = A\mathbf{x}_0 - \mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ = 4 \end{bmatrix}$$

We have

$$\mathbf{x}_1 = \mathbf{x}_0 - \alpha_0 f'(\mathbf{x}_0)$$

where

$$\alpha_0 = \frac{\mathbf{r}_0^T \mathbf{r}_0}{\mathbf{r}_0^T A \mathbf{r}_0}$$

We have

$$\mathbf{r}_0 = -f'(\mathbf{x}_0) = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

and thus, doing the computation by your graphical calculator as that's much faster,

$$\alpha_0 = 0.1818$$

and thus

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0.1818 \cdot \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.7273 \\ 0.7273 \end{bmatrix}$$

and thus answer D is correct.

1.28. Question 10

The correct answer is A: this is a rather hard question. First of all, if the optimum is found in one step, we must have

$$f'(\mathbf{x}_1) = A\mathbf{x}_1 - \mathbf{b} = 0$$

Now, $\mathbf{x}_1 = \mathbf{x}_0 - \alpha \mathbf{r}_0$, and thus we must have

$$A(\mathbf{x}_0 - \alpha f'(\mathbf{x}_0)) - \mathbf{b} = 0$$

where

$$\mathbf{f}_0' = A\mathbf{x}_0 - \mathbf{b}$$

and thus

$$\begin{aligned} A(\mathbf{x}_0 - \alpha(A\mathbf{x}_0 - \mathbf{b})) - \mathbf{b} &= 0 \\ A\mathbf{x}_0 - A\alpha A\mathbf{x}_0 + \alpha A\mathbf{b} + \mathbf{b} &= 0 \end{aligned}$$

Now the magic happens, as we can rewrite this a fair bit. Remember that $I\mathbf{b} = \mathbf{b}$ where I is the identity matrix. Similarly, $IA = A$, meaning we are allowed to write

$$\begin{aligned} IA\mathbf{x}_0 - \alpha A\mathbf{x}_0 + \alpha A\mathbf{b} - I\mathbf{b} &= 0 \\ (I - \alpha A)(A\mathbf{x}_0 - \mathbf{b}) &= 0 \end{aligned}$$

Now, *assume* answer A is correct. Then we'd have $A = kI$, where k is some scalar, $k \in \mathbb{R}$. We then have

$$\begin{aligned} (I - \alpha A)(A\mathbf{x}_0 - \mathbf{b}) &= 0 \\ (I - \alpha kI)(kI\mathbf{x}_0 - \mathbf{b}) &= 0 \\ I(1 - \alpha k)(k\mathbf{x}_0 - \mathbf{b}) &= 0 \\ (1 - \alpha k)(k\mathbf{x}_0 - \mathbf{b}) &= 0 \end{aligned}$$

Now, this may seem rather worthless, because what could we deduce from this? Well, let's, just for fun, actually deduce the value of α:

$$\alpha = \frac{\mathbf{r}_0^T \mathbf{r}_0}{\mathbf{r}_0^T A \mathbf{r}_0}$$

However, again we have $A = kI$, and thus

$$\begin{aligned} \alpha &= \frac{\mathbf{r}_0^T \mathbf{r}_0}{\mathbf{r}_0^T A \mathbf{r}_0} \\ \alpha &= \frac{\mathbf{r}_0^T \mathbf{r}_0}{\mathbf{r}_0^T kI \mathbf{r}_0} = \frac{\mathbf{r}_0^T \mathbf{r}_0}{k\mathbf{r}_0^T \mathbf{r}_0} = \frac{1}{k} \end{aligned}$$

and thus we get

$$\begin{aligned} (1 - \alpha k)(k\mathbf{x}_0 - \mathbf{b}) &= 0 \\ \left(1 - \frac{k}{k}\right)(k\mathbf{x}_0 - \mathbf{b}) &= 0 \\ (1 - 1)(k\mathbf{x}_0 - \mathbf{b}) = 0 \cdot (k\mathbf{x}_0 - \mathbf{b}) &= 0 \end{aligned}$$

and thus this equation is always satisfied, whatever the value of $\mathbf{x}_0$ is! Beautiful result: we have proven that as long as A is a multiple of the identity matrix, the solution converges to the exact minimum in 1 iteration. The correct answer is A.

1.29. Question 11

The correct answer is H: let's just follow the algorithm. First, we compute

$$\begin{aligned} f(0,0) &= 3 \cdot 0^2 + 0^2 - 4 \cdot 0 + 1 = 1 \\ f(.5,.5) &= 3 \cdot 0.5^2 + 0.5^2 - 4 \cdot 0.5 + 1 = 0 \\ f(.5,0) &= 3 \cdot 0.5^2 + 0^2 - 4 \cdot 0.5 + 1 = -0.25 \end{aligned}$$

Ordering them from lowest to highest clearly gives $x_1 = (.5, 0)$, $x_2 = (.5, .5)$ and $x_3 = (0, 0)$. Then, x_0 becomes

$$x_0 = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}\left(\begin{bmatrix} 0.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right) = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

Then, x_R equals

$$x_R = x_0 + (x_0 - x_3) = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} + \left(\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

for which

$$f(1,0.5) = 3\cdot 1^2 + 0.5^2 - 4\cdot 1 + 1 = 0.25$$

Thus, $f(x_e) > f(x_2)$ and we need to CONTRACT. For this, we get

$$x_C = x_0 + \frac{1}{2}(x_0 - x_3) = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0.75 \\ 0.375 \end{bmatrix}$$

$f(0,0) = 1$ is then replaced by

$$f(0.75, 0.375) = 3\cdot 0.75^2 + 0.375^2 - 4\cdot 0.75 + 1 = -0.172$$

Thus, we are left with the values $-0.25, -0.172, 0.0$ and thus answer H is correct.

1.30. Question 12

The correct answer is B: should be pretty logical, but if you want a more definite explanation: this means that $f(x_R)$ will never be smaller than $f(x_1)$ and thus we will never EXPAND. CONTRACTing will reduce the area of the simplex, thus answer B is correct.

1.31. Question 13

The correct answer is B: $\tilde{x}_i$ are values that minimize $g(x)$. For this, it is required that $g'(x_i) = 0$ and $g''(x_i) > 0$, otherwise it wouldn't be a minimum.