Part I

Module 3

1.1. Question 1

The correct answer is D: for a spline of degree d, $s, s', ..., s^{(d-1)}$ are all continuous. For a cubic spline, $d = 3$ thus everything up to and including the second derivative is continuous. In case you wonder, but aren't cubic formulas infinitely differentiable? Cubic function themselves are, but splines, which consists of several cubic functions on each subinterval, are not: there will be discontinuities in the slope of the slope of the slope at the data points themselves.

1.2. Question 2

The correct answer is B: A is not a requirement, C is not, and D is not. Only is required that the data points are in ascending or descending order because otherwise everything rather obviously gets problematic.

1.3. Question 3

The correct answer is C. We have five things to determine basically: the values of y at $x = 4$ and $x = 6$, and b, c and d. Finding $y(x = 4)$ and $y(x = 6)$ is easy:

$$\begin{aligned} f(4) &= -2 \cdot 4^2 + 14 \cdot 4 - 9 = 15 \\ f(6) &= 25 \cdot 6^2 - 303 \cdot 6 + 928 = 10 \end{aligned}$$

That's nice, but actually we don't even need them: remember that for a quadratic spline, the first derivative should be continuous, i.e. $f'(x) = f'(x)$ every data point, irrespective of which of the two interpolants. For $x = 4$, we have two derivatives of importance. First, the one for $x \in [2, 4]$:

$$\begin{aligned} f'(x) &= -4x + 14 \\ f'(4) &= -4 \cdot 4 + 14 = -2 \end{aligned}$$

and for $x \in [4, 6]$:

$$\begin{gathered} f'(x) = 2bx + c \\ f'(4) = 2b \cdot 4 + c = 8b + c \end{gathered}$$

so that our first equation becomes $8b + c = -2$. Similarly, we have for $x = 6$ that for the function for $x \in [6, 7]$:

$$\begin{aligned} f'(x) &= 50x - 303 \\ f'(6) &= 50 \cdot 6 - 303 = -3 \end{aligned}$$

and for $x \in [4, 6]$:

$$f'(6) = 2b \cdot 6 + c = 12b + c$$

so that we have our system of equations

$$\begin{aligned} 8b + c &= -2 \\ 12b + c &= -3 \end{aligned}$$

Solving this with your graphical calculator results in $c = 0$. Thus, the correct answer is C (note that we actually never used the values of y at $x = 4$ or $x = 6$).

1.4. Question 4

The correct answer is D: remember that for a spline of degree d, the derivatives up to $d-1$ should be continuous. For A, $d=1$, so only s needs to be continuous. We clearly see that for both functions, $f(1)=2$ and thus s is continuous, and thus A is a spline. For B, $d=2$, so both $f(1)$ and $f'(x)$ should be the same result for both functions. We clearly have $f(1)=2$ for both, and simply algebra also shows $f'(1)=2$. Thus, B is a spline. C has $d=3$, and thus all of $f(1)$, $f'(1)$ and $f''(1)$ need to be the same value for both functions. You can easily verify that $f(1)=1$, $f'(1)=2$ and $f''(1)=2$ as well. For D, $d=3$ as well thus again all of $f(1)$, $f'(1)$ and $f''(1)$ need to be the same value for both functions. Although we have $f(1)=1$ for both, $f'(1)=2$ for the first one and $f'(1)=0$ for the second one. Thus, D is not a spline, as it is degree 3, but s' is not continuous. Note that this requirement only follows from the fact that D is cubic: for A, s' is not continuous as well, but A is linear so s' is not required to be cubic.

1.5. Question 5

The correct answer is A: note in this question, they use x' and y' to denote the local coordinates (where I used x and y for), and they use x and y to denote the real coordinates (where I used η and ξ for). Now, by inspection, we see

$$x' = 2x-3, \quad y' = y+2$$

You don't see it that quickly or you just want a method you can rely on? Essentially, we have must transform (for x) $x=1$ to $x'=-1$ and $x=2$ to $x'=1$. We know that our formula must look like $x'=ax+b$. Thus, we get the system of equations

$$\begin{aligned} -1 &= a\cdot 1+b \\ 1 &= a\cdot 2+b \end{aligned}$$

Solving this leads to $a=2$ and $b=-3$ and thus $x'=2x-3$. Similarly, for y, we must transform $y=-3$ to $y'=-1$ and $y=-1$ to $y'=1$. Thus:

$$\begin{aligned} -1 &= c\cdot -3+d \\ 1 &= c\cdot -1+d \end{aligned}$$

so that $c=1$ and $d=2$, so that $y'=y+2$. In any case, these two formulas can be applied to the point $(x=5/3, y=-4/3)$, as plugging this in leads to $(x'=1/3, y'=2/3)$, so that we have

$$\begin{aligned} \phi(1/3,2/3) &= \frac{1.3}{4}\left(1-\frac{1}{3}\right)\left(1-\frac{2}{3}\right)+\frac{1.5}{4}\left(1+\frac{1}{3}\right)\left(1-\frac{2}{3}\right) \\ &\quad +\frac{0.3}{4}\left(1+\frac{1}{3}\right)\left(1+\frac{2}{3}\right)+\frac{0.2}{4}\left(1-\frac{1}{3}\right)\left(1+\frac{2}{3}\right) \\ &= 0.4611 \end{aligned}$$

so that answer A is correct.

1.6. Question 6

The correct answer is E: again, we use the same formula as in the previous question. This time, we have

$$x' = 2x-1, \quad y' = y-1$$

so that $x' = 2\cdot 2/3 - 1 = 1/3$ and $y' = 4/3 - 1 = 1/3$, and thus

$$\begin{aligned}\phi\left(\frac{1}{3},\frac{1}{3}\right) &= \frac{1.7}{4}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3}\right)+\frac{3.9}{4}\left(1+\frac{1}{3}\right)\left(1-\frac{1}{3}\right) \\ &\quad +\frac{0.6}{4}\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3}\right)+\frac{1.6}{4}\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right) \\ &= 1.678\end{aligned}$$

and thus answer E is correct.

1.7. Question 7

The correct answer is F: I said it would be easy and it really is. The linear interpolant looks like $g(x,y) = a_0 + a_1 x + a_2 y$. As $g(0,0) = 2$, this immediately means that $a_0 = 2$ (should be really obvious). If we then compare $i = 2$ and $i = 3$, we see that if we increase y_i by 1, f_i decreases by 2; from this, we deduce $a_2 = -2$. Then, comparing $i = 1$ and $i = 2$: we think, $f_1 = 2$, and between $i = 1$ and $i = 2$, y has increased by 1, so f_2 should have decreased to 0 as well. However, apparently, the increase of x by 1 has increased the value of f_2 by 3 (relative to the 0 you'd expect if y would be the only influencing parameter), this means that $a_1 = 3$. Thus, our final formula is

$$g(x,y) = 2 + 3x - 2y$$

and thus

$$g\left(\frac{2}{3},1\right) = 2 + 3\cdot\frac{2}{3} - 2\cdot 1 = 2$$

and thus answer F is correct.

1.8. Question 8

The correct answer is C. It is helpful to first plug in the values you have to interpolate to get the table below.

i	1	2	3
x_i	0	1	1
y_i	0	0	1
f_i	0	1	1.41

We then see that if we increase x by 1, f increases by 1 as well. If we increase y by 1, f increases by 0.41. At $x = 0$ and $y = 0$, $f = 0$, so the formula becomes

$$g(x,y) = x + 0.41y$$

so that

$$g\left(\frac{2}{3},\frac{1}{2}\right) = \frac{2}{3} + 0.41\cdot\frac{1}{2} = 0.8717$$

and thus answer C is correct.

1.9. Question 9

The correct answer is C: again, we simply use

$$g(\alpha,\delta) = a_0 + a_1\alpha + a_2\delta$$

and we immediately recognized $a_0 = 0.20$. Furthermore, $a_1 = (1.00 - 0.20)/(8-0) = 0.10$ and $a_2 = (1.16-1.00)/(4-0) = 0.04$. Thus,

$$g(4,1) = 0.20 + 0.10 \cdot 4 + 0.04 \cdot 1 = 0.64$$

and thus answer C is correct.

1.10. Question 10

The correct answer is D: for this question, remember the formula

$$A_{ij} = \Phi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right)$$

In other words, in the function $\phi(r) = r$, r is the distance between (x_2, y_2) and (x_3, y_3). You can compute this straightforwardly:

$$r = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} = \sqrt{(2-1)^2 + (1-0)^2} = 1.41$$

so that $\phi(r) = 1.41$ for $A_{2,3}$ and thus answer D is correct.

1.11. Question 11

The correct answer is B: again, it is quintessential to remember

$$A_{ij} = \Phi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right)$$

So, for example, to determine the entry A_{23}, you find the distance between $\mathbf{x}_2$ and $\mathbf{x}_3$ (where $\mathbf{x}_i = (x_i, y_i)$), and then plug this into $\phi(r) = e^{-r}$. Note that $\mathbf{A}$ is symmetric: $A_{23} = A_{32}$, meaning we only have to compute the values of the topright half of the matrix. Furthermore, for the diagonal, i.e. A_{11}, A_{22} and A_{33}, we must take the distance between the same vectors $\mathbf{x}_i$, which obviously equals 0. We then have $\phi(0) = e^{-0} = 1.000$, so the diagonal entries of the matrix are all 1.000. We only have to compute A_{12}, A_{13} and A_{23} then:

$$\begin{aligned}
r_{12} &= \sqrt{(1-0)^2 + (0-0)^2} = 1 \\
A_{12} &= \phi(1) = e^{-1} = 0.368 \\
r_{13} &= \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2} \\
A_{13} &= \phi\left(\sqrt{2}\right) = e^{-\sqrt{2}} = 0.243 \\
r_{23} &= \sqrt{(1-1)^2 + (1-0)^2} = 1 \\
A_{23} &= \phi(1) = e^{-1} = 0.368
\end{aligned}$$

so that the matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} 1.000 & 0.368 & 0.243 \\ 0.368 & 1.000 & 0.368 \\ 0.243 & 0.368 & 1.000 \end{bmatrix}$$

1.12. Question 12

The correct answer is D. We have two coefficients, so we obtain a two-by-two matrix equation of

$$\begin{bmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_1, \phi_2 \rangle \\ \langle \phi_2, \phi_1 \rangle & \langle \phi_2, \phi_2 \rangle \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix} = \begin{bmatrix} \langle f, \phi_1 \rangle \\ \langle f, \phi_2 \rangle \end{bmatrix}$$

where

$$\left\langle \phi_j, \phi_k \right\rangle = \sum_{i=1}^{N} \phi_j(\mathbf{x}_i)\phi_k(\mathbf{x}_i)$$

So, since we have

$$\begin{aligned}\phi_1(x) &= x\\ \phi_2(x) &= 1\end{aligned}$$

we obtain

$$\begin{aligned}\left\langle \phi_1, \phi_1 \right\rangle &= \sum_{i=1}^{N} \phi_j(\mathbf{x}_i)\phi_k(\mathbf{x}_i) = \sum_{i=1}^{5} x_i \cdot x_i = \sum_{i=1}^{5} x_i^2\\ &= 0.1^2 + 0.2^2 + 0.4^2 + 0.5^2 + 0.8^2 = 1.1\\ \left\langle \phi_1, \phi_2 \right\rangle &= \sum_{i=1}^{N} \phi_j(\mathbf{x}_i)\phi_k(\mathbf{x}_i) = \sum_{i=1}^{5} x_i \cdot 1 = \sum_{i=1}^{5} x_i\\ &= 0.1 + 0.2 + 0.4 + 0.5 + 0.8 = 2\\ \left\langle \phi_2, \phi_2 \right\rangle &= \sum_{i=1}^{N} \phi_j(\mathbf{x}_i)\phi_k(\mathbf{x}_i) = \sum_{i=1}^{5} 1 \cdot 1 = \sum_{i=1}^{5} 1\\ &= 1 + 1 + 1 + 1 + 1 = 5\end{aligned}$$

where it is noted that $\left\langle \phi_1, \phi_2 \right\rangle = \left\langle \phi_2, \phi_1 \right\rangle$. Similarly,

$$\begin{aligned}\left\langle f, \phi_1 \right\rangle &= \sum_{i=1}^{N} f(\mathbf{x}_i)\phi_k(\mathbf{x}_i) = \sum_{i=1}^{5} y_i \cdot x_i = 21 \cdot 0.1 + 18 \cdot 0.2 + 15 \cdot 0.4 + 15 \cdot 0.5 + 12 \cdot 0.8 = 28.8\\ \left\langle f, \phi_1 \right\rangle &= \sum_{i=1}^{N} f(\mathbf{x}_i)\phi_k(\mathbf{x}_i) = \sum_{i=1}^{5} y_i \cdot 1 = 21 + 18 + 15 + 15 + 12 \cdot 0.8 = 81\end{aligned}$$

and thus we solve

$$\begin{aligned}\begin{bmatrix} 1.1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 28.8 \\ 81 \end{bmatrix}\\ \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} -12 \\ 21 \end{bmatrix}\end{aligned}$$

so answer D is correct.

1.13. Question 13

The correct answer is B. This is very similar to the previous question, but now we use different symbols, and instead of $\phi_1(x) = x$ we use $\phi_1(\Delta p) = \sqrt{\Delta p}$. Analogous to the previous question, we get

$$\left\langle \phi_1, \phi_1 \right\rangle = \sum_{i=1}^{4} \Delta p_i = 0.1$$

$$\left\langle \phi_1, \phi_2 \right\rangle = \sum_{i=1}^{4} \sqrt{\Delta p_i} = 0.6146$$

$$\left\langle \phi_2, \phi_2 \right\rangle = \sum_{i=1}^{4} 1 = 4$$

where it is noted that $\left\langle \phi_1, \phi_2 \right\rangle = \left\langle \phi_2, \phi_1 \right\rangle$. Similarly,

$$\left\langle f, \phi_1 \right\rangle = \sum_{i=1}^{4} v_i \cdot \sqrt{\Delta p_i} = 45.79$$

$$\left\langle f, \phi_1 \right\rangle = \sum_{i=1}^{4} v_i \cdot 1 = 277$$

and thus we solve

$$\begin{bmatrix} 0.1 & 0.6146 \\ 0.6146 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 45.79 \\ 277 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 581.05 \\ -20.03 \end{bmatrix}$$

and thus answer B is correct.

1.14. Question 14

The correct answer is D. In analogous fashion to the previous two questions, but with $\phi_1(x) = 1/(1+e^x)$ and $\phi_2(x) = x$, we now solve the normal equation

$$\begin{bmatrix} 2.4542 & -10.343 \\ -10.343 & 144 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6.921 \\ -24.4 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3.02 \\ 0.05 \end{bmatrix}$$

1.15. Question 15

The correct answer is A. In analogous fashion to the previous two questions, but with $\phi_1(x) = 1/(1+e^x)$ and $\phi_2(x) = x$, we now solve the normal equation

$$\begin{bmatrix} 0.43795 & 17.7981 \\ 17.7981 & 725 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 335.76 \\ 13575 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2436 \\ -41.078 \end{bmatrix}$$

Note that the answer is quite sensitive to rounding in between - reasonably you should expect to get within 1% of the correct answer (so 2436 ± 24).

1.16. Question 16

The correct answer is D: A, B and C are all just wrong. If $N = I$, you are essentially interpolating, because you have enough coefficients to make the regression function go exactly through all of the data points. It is not required that the data is not noisy.

1.17. Question 17

The correct answer is C. First of all, D is the worst option, as it then does not passes through the points. A linear spline is not smooth at all, as it's not even once continuously differentiable. A fifth order polynomial will be very smooth, but will very likely be longer than quadratic splines due to interpolation errors. So, answer C is correct, though the question is really wrongly posed because whatever you answer should actually be correct from a semantic point of view.

Part II

Module 4

1.18. Question 1

The correct answer is C: rather clearly, 2 is a backward difference and 3 is a forward difference. 1 looks an awful lot like the central difference, but they forgot to divide by $2h$ instead of simply h.

1.19. Question 2

The correct answer is B: although the third one, the central differences formula, should generally be preferred, for $f(0.0)$, we don't have access to $f(0.0-h)$, thus it is impossible to apply that one. Thus, we must use the forward difference formula:

$$f'(0.0) = \frac{f(0.0+0.3)-f(0.0)}{0.3} = \frac{1.3-1.0}{0.3} = 1.0$$

so answer B is correct.

1.20. Question 3

The correct answer is D: again, we run into the problem of not having access to $f(1.2+h)$, so this time, we must use the backwards differences scheme:

$$f'(1.2) = \frac{f(1.2)+f(1.2-0.3)}{0.3} = \frac{3.3-2.4}{0.3} = 3.00$$

so the correct answer is D.

1.21. Question 4

The correct answer is A: you could see this coming from miles away: we need to apply the central differences one here:

$$f'(0.3) = \frac{f(0.3+0.3)-f(0.3-0.3)}{2\cdot 0.3} = \frac{1.8-1.0}{0.6} = 1.33$$

so answer A is correct.

1.22. Question 5

The correct answer is B: it's basically the same story as the previous few questions: again, we look for the scheme that we can use with the largest relation with h; in this case, that's clearly the fourth equation:

$$f'(2.1) = \frac{1}{2\cdot 0.1}\left[-3\cdot f(2.1)+4\cdot f(2.1+0.1)-f(2.1+2\cdot 0.1)\right] = \frac{1}{0.2}\left[-3\cdot 1.7+4\cdot 1.3-1.1\right] = -5$$

and thus answer B is correct.

1.23. Question 6

The correct answer is A: now we can even use the sixth equation:

$$\begin{aligned} f'(2.3) &= \frac{1}{12\cdot 0.1}\left[f(2.3-2\cdot 0.1)-8\cdot f(2.3-0.1)+8\cdot f(2.3+0.1)-f(2.3+2\cdot 0.1)\right] \\ &= \frac{1}{1.2}\left[1.7-8\cdot 1.3+8\cdot 0.9-0.7\right] = -1.83 \end{aligned}$$

so answer A is correct.

1.24. Question 7

The correct answer is B: we must now use the fifth formula:

$$f'(2.6) = \frac{1}{2 \cdot 0.1}\left[3f(2.6) - 4f(2.6-0.1) + f(2.6 - 2\cdot 0.1)\right] = \frac{1}{0.2}\left[3\cdot 0.6 - 4\cdot 0.7 + 0.9\right] = -0.5$$

so answer B is correct.

1.25. Question 8

The correct answer is B. To answer this question, we first find the Taylor series expansion for the functions, including terms up to h^3:

$$\begin{aligned} f(x+3h) &= f(x) + 3hf'(x) + 9h^2 f''(x) + 27h^3 f'''(x) + \cdots \\ f(x+2h) &= f(x) + 2hf'(x) + 4h^2 f''(x) + 8h^3 f'''(x) + \cdots \\ f(x+h) &= f(x) + hf'(x) + h^2 f''(x) + 4h^3 f'''(x) + \cdots \end{aligned}$$

Now, plug this into the given difference formula:

$$\begin{aligned} f'(x) &\approx \frac{1}{2h}\Big(2\cdot\left[f(x) + 3hf'(x) + 9h^2 f''(x) + 27h^3 f'''(x)\right] \\ &\quad -7\cdot\left[f(x) + 2hf'(x) + 4h^2 f''(x) + 8h^2 f'''(x)\right] \\ &\quad +10\cdot\left[f(x) + hf'(x) + h^2 f''(x) + h^3 f'''(x)\right] - 5f(x)\Big) \\ f'(x) &\approx \frac{1}{2h}(2-7+10-5)\cdot f(x) + \frac{1}{2h}\cdot(2\cdot 3h - 7\cdot 2h + 10h) f'(x) \\ &\quad + \frac{1}{2h}\cdot\left(2\cdot 9h^2 - 7\cdot 4h^2 + 10h^2\right) f''(x) + \frac{1}{2h}\cdot\left(2\cdot 27h^3 - 7\cdot 8h^3 + 10h^3\right) f'''(x) \end{aligned}$$

This can be simplified to

$$f'(x) \approx f'(x) + 4h^2 f'''(x) + \cdots$$

so that the order of the truncation error is clearly $O\left(h^2\right)$, and thus answer B is correct.

1.26. Question 9

The correct answer is D: this is simply remembering and applying the formulas. For forward difference:

$$f'(0) = \frac{f(0+0.5) - f(0)}{0.5} = \frac{0.5^2 - 0^2}{0.5} = 0.50$$

For the central differences:

$$f'(0) = \frac{f(0+0.5) - f(0-0.5)}{2\cdot 0.5} = \frac{(-0.5)^2 - 0.5^2}{1.0} = 0.00$$

and thus answer D is correct.

1.27. Question 10

The correct answer is H: this is most easily done by comparing with the derivation for the forward difference error: this time, instead of

$$f'(x) = \frac{F(x+h)+\epsilon(x+h)-F(x)-\epsilon(x)}{h} - \frac{1}{2}hf''(\xi)$$

we have

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} - \frac{h^2}{6}f^{(3)}(\xi), \quad x-h<\xi<x+h$$

Thus, it makes a lot sense that we can write

$$\begin{aligned}\left|f'(x) - \frac{F(x+h)-F(x-h)}{h}\right| &\leq \left|\frac{\epsilon(x+h)-\epsilon(x-h)}{2h}\right| \leq \left|\frac{\epsilon(x+h)-\epsilon(x-h)}{2h} - \frac{h^2}{6}f'''(\xi)\right| \\ e(h) &= 2\cdot\frac{\epsilon}{2h} + \frac{M}{6}h^2 = \frac{\epsilon}{h} + \frac{M}{6}h^2\end{aligned}$$

This means that we have

$$\begin{aligned}\frac{de}{dh} &= -\frac{\epsilon}{h^2} + \frac{M}{3}h = 0 \\ \frac{\epsilon}{h^2} &= \frac{M}{3}h \\ h^3 &= \frac{3\epsilon}{M} \\ h &= \sqrt[3]{\frac{3\epsilon}{M}} = \sqrt[3]{\frac{3\cdot5\cdot10^{-6}}{0.7}} \approx 0.028\end{aligned}$$

and thus answer H is correct.

1.28. Question 11

The correct answer is A: so we must place 2 equidistant nodes within the interval [0,1], not using $x_0 = 0$ and $x_1 = 1$. Rather obviously, this means we have $x_0 = 1/3$ and $x_1 = 2/3$ (if you'd had used 4 nodes, you'd had gotten $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$ and $x_3 = 1$; then just get rid of $x_0 = 0$ and $x_3 = 1$). The weights then equal

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}^{-1} \begin{bmatrix} \int_0^1 dx \\ \int_0^1 xdx \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

and thus

$$Q(f;0,1) = \frac{1}{2}\cdot\sqrt{1+\frac{1}{3}} + \frac{1}{2}\cdot\sqrt{1+\frac{2}{3}} = 1.2228$$

and thus answer A is correct.

1.29. Question 12

The correct answer is D. You just have to try and see whether it approximates $\int_{-1}^{1} dx$, $\int_{-1}^{1} x dx$, $\int_{-1}^{1} x^2 dx$ exactly. For example, we have for $f(x) = x^2$:

$$\int_{-1}^{1} x^2 dx = 0.66667$$

$$\frac{2}{27} \cdot \left[8 \cdot \left(-\frac{3}{4}\right)^2 + 11 \cdot 0^2 + 8 \cdot \left(\frac{3}{4}\right)^2 \right] = 0.66667$$

so it still approximates exactly for a quadratic polynomial. For $f(x) = x^3$:

$$\int_{-1}^{1} x^3 dx = 0$$

$$\frac{2}{27} \cdot \left[8 \cdot \left(-\frac{3}{4}\right)^3 + 11 \cdot 0^3 + 8 \cdot \left(\frac{3}{4}\right)^3 \right] = 0$$

and thus it still is exact for a cubic polynomial. However, for $f(x) = x^4$:

$$\int_{-1}^{1} x^4 dx = 0.4$$

$$\frac{2}{27} \cdot \left[8 \cdot \left(-\frac{3}{4}\right)^4 + 11 \cdot 0^4 + 8 \cdot \left(\frac{3}{4}\right)^4 \right] = 0.375$$

and thus it is no longer exact for polynomials of degree higher than 3, thus the degree of precision is 3, and thus answer D is correct. Do note: you have to start these questions from the lowest number, i.e. you have to start at comparing $\int_{-1}^{1} dx$ and not at $\int_{-1}^{1} x^7 dx$: x^7 is actually still exactly approximated and may have caused you to believe, so then it must approximate x^6, x^5, etc. also exactly. However, it's just coincidence that x^7 is exactly approximated (well, sort of coincidence). You *must* start at the bottom and then check which is the first smallest degree where it goes wrong.

1.30. Question 13

The correct answer is B: just plug in

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ x_0 & x_1 \end{bmatrix}^{-1} \begin{bmatrix} \int_{-1}^{1} dx \\ \int_{-1}^{1} x dx \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and thus $w_0 = w_1 = 1$ and thus answer B is correct. In general, the sum of the weights should equal the length of the interval. If this is not the case, your degree of precision isn't even 0 which is just embarrassingly bad to be honest.

1.31. Question 14

The correct answer is B: again, the sum of the weights must equal the length of the interval, which is clearly only the case when $w_0 = w_1 = 1$. If you're like, nahh I wanna spend more than five seconds on this question, you can also derive it like this:

$$\begin{bmatrix} 1 & 1 \\ -a & a \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} dx \\ \int_{-1}^{1} x dx \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

so we have $w_0 + w_1 = 2$ and $-aw_0 + aw_1 = 0$, so that from the second one we deduce that $w_0 = w_1$ and from the first one then $w_0 = w_1 = 1$, so the correct answer is B.

1.32. Question 15

The correct answer is B: note that we do not have to calculate the weights this time actually, so what do we do? Well, we apparently must have the equality

$$\begin{aligned} \int_{-1}^{1} x^3 dx &= (-a)^3 + a^3 \\ 0 &= -a^3 + a^3 = 0 \end{aligned}$$

so this equality is satisfied, no matter the value of a. However, if we require a degree of precision of 3, then we must also have

$$\begin{aligned} \int_{-1}^{1} x^2 dx &= (-a)^2 + a^2 \\ \frac{2}{3} &= 2a^2 \\ a &= \frac{1}{\sqrt{3}} \end{aligned}$$

Yes, you can now also check whether it works for a polynomial of degree 1 or even 0, but if it didn't, whatcha gonna do about it? You can't change a any more, so answer B is correct.

1.33. Question 16

The correct answer is C: there are two ways to do this one. First of all, which is the non-stupid way to do it, is realize that the trapezoidal rule will be able to interpolate polynomials with monomials up to degree 2 exactly (i.e., x^2, y^2 and xy, and everything below that). Thus, the integral can be evaluated exactly, and we know that the area of a circle with radius 1 is simply equal to $\pi = 3.1416$, thus answer C is correct. Alternatively, you can follow the hint. We have the transformations

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. *Note that this is a rectangle*: we draw r on the horizontal axis, and θ on the vertical axis, and draw this area: we see directly that this is a rectangle, with width 1 and height 2π.

What has happened with the integral? This has become

$$\iint_{\circ} 1\cdot dxdy = \int_0^1\int_0^{2\pi} 1\cdot|J(x,y)|\cdot d\theta dr$$

where

$$|J(x,y)| = \left|\det\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}\right| = \left|\det\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}\right| = \left|r\cos^2\theta - -r\sin^2\theta\right| = r$$

so we have to integrate

$$\int_0^1\int_0^{2\pi} r\cdot d\theta dr$$

Numerically, this can be written as

$$\int_0^1 rdr\int_0^{2\pi} d\theta = \sum_{i=1}^{s_1} w_i \sum_{j=1}^{s_2} v_{ji} r_i$$

As we use the trapezoidal rule, we have for w_i, as we have the interval $[0,1]$, that $w_1 = w_2 = 1/2$ and the nodes $r_1 = 0$ and $r_2 = 1$. Furthermore, for v_{ji}, we have then that for both values of r, θ ranges between 0 and 2π, and thus we must use $v_{11} = v_{12} = v_{21} = v_{22} = \pi$, so that we get

$$I = \frac{1}{2}\cdot(\pi\cdot 0 + \pi\cdot 0) + \frac{1}{2}\cdot(\pi\cdot 1 + \pi\cdot 1) = \pi$$

You see that it is honestly just so much better to realize what the power is of degree of precision, as this sum is utterly confusing to do by hand.

1.34. Question 17

The correct answer is C: just plug the exact integral into your graphical calculator, and you see that $I = 2.67$. For the Trapezoidal rule, it is slightly misleading: the trapezoidal rule only uses two nodes, but three nodes are given. This implies that we have to use the *composite* trapezoidal rule. This means that we get

$$I_3 = \frac{1--1}{2}\left(\frac{1}{2}f(-1) + f(0) + f(1)\right) = \frac{1}{2}\cdot\left[(-1)^2+1\right) + \left[0^2+1\right] + \frac{1}{2}\cdot\left[1^2+1\right] = 3.00$$

and thus answer C is correct.

1.35. Question 18

The correct answer is G: just plug the exact integral into your calculator; it'll equal $I = 2.35$. The approximation simply becomes

$$Q(e^x;-1,1) = 0.555556\cdot e^{-0.774597} + 0.8888889\cdot e^{0.00000} + 0.555556\cdot e^{0.774597} = 2.35$$

and thus answer G is correct.

1.36. Question 19

The correct answer is E: the exact integral can be easily computed with your graphical calculator, and equals $I = 1.72$. Furthermore, note that the Gaussian Quadrature nodes and weights are based on the interval $[-1,1]$, but we need to go to $[0,1]$. That means that the nodes and weights respectively become

$$\begin{aligned}
w_1' = w_3' &= \frac{1-0}{1--1}\cdot 0.555556 = 0.277778 \\
w_2' &= \frac{1-0}{1--1}\cdot 0.888889 = 0.444445 \\
x_1' &= \frac{1-0}{1--1}\cdot (-0.774597--1)+0 = 0.1127 \\
x_2' &= \frac{1-0}{1--1}\cdot (0.00000--1)+0 = 0.5 \\
x_3' &= \frac{1-0}{1--1}\cdot (0.774597--1)+0 = 0.8873
\end{aligned}$$

and therefore

$$I_3 = Q(e^x;0,1) = 0.277778\cdot e^{0.1127} + 0.444445\cdot e^{0.5} + 0.277778\cdot e^{0.8873} = 1.72$$

so answer E is correct.