Applied Numerical Analysis – AE2220-I – Quiz #3

Modules 5 and 6 – April, 2022

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable "Answer sheet".
- Use a black or blue pen to fill in the answer form.
- Write your name and student number on the answer sheet.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- Graphical calculators of all kinds are not allowed; a scientific calculator is recommended.
- This quiz has **10 questions** and **4 pages (2 sheets)** in total.

Numerical solution of ODEs

Question 1 Consider the equation of the harmonic damped oscillator with an external time dependent forcing:

$$x'' + x' + x = \sin(t^2)$$

with the initial-conditions $x(0) = 0$ and $x'(0) = 2$. Applying forward-Euler method with a timestep $\Delta t = \frac{1}{2}$, what is the position x_1 and the velocity x_1' after one iteration?

A: Unanswered
B: $x_1 = x_1' = 1.0$
C: $x_1 = x_1' = 0.5$
D: $x_1 = x_1' = 0.0$
E: $x_1 = x_1' = -0.5$
F: $x_1 = x_1' = -1.0$

Question 2 Consider the ODE $u' = -2cu$ with initial condition $u(0) = 1$ and constant $c > 0$. Using backward-Euler and $\Delta t = \frac{1}{2}$, what is the approximation of $u(4)$?

A: Unanswered
B: $(1 + c)^{-8}$
C: $(1 + c/2)^8$
D: $(1 - c)^{-8}$
E: $(1 - c/2)^8$
F: $(c)^{-8}$
G: $(c/2)^8$

Question 3 We're concerned with solving the ODE $u' = f(u)$. Consider the scheme:

$$u_{i+1} = u_i + \frac{3}{2}\Delta t f(u_i) - \frac{1}{2}\Delta t f(u_{i-1}),$$

where u_i and u_{i-1} are known, and u_{i+1} is unknown. What is the *local*-truncation error of this scheme? [Hint: Use Taylor expansions of all terms; for the term $f(u_{i-1})$ expand first u_{i-1}, then f of the first two terms in the expansion.]

A: Unanswered
B: Δt^{-1}
C: Δt^0
D: Δt^1
E: Δt^2
F: Δt^3
G: Δt^4
H: Δt^5

Question 4 **(Module 5 - ODEs)** Consider the ODE $y' = -ay$ with $a > 0$. Using the forward-Euler scheme we can write:

$$y_{i+1} = y_i + h(-ay_i).$$

By writing the solution at y_{i+1} in terms of the initial condition y_0, or otherwise, decide for what range of stepsize h is the scheme stable for this equation?

A: Unanswered
B: Never stable
C: Always stable
D: $h > 0$
E: $0 < h < 1/a$
F: $0 < h < a$
G: $0 < h < 2/a$
H: $0 < h < 2a$

Question 5 Consider the following multi-stage time-stepping scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(k_1 + k_2),$$
$$k_1 = f(y_n),$$
$$k_2 = f(y_n + hk_1).$$

This scheme is known as the 2nd-order Runge-Kutta method. Assume for simplicity that $f(y) = cy$, with c a constant. Let $z = ch$. For what values of z is the scheme stable?

A: Unanswered
B: $|1 + z| < 1$
C: $|1 - z| < 1$
D: $|1 + z + \frac{1}{2}z^2| < 1$
E: $|1 + z + \frac{1}{2}z^2 + \frac{1}{12}z^3| < 1$
F: $|1 + z + z^2| < 1$
G: $|1 + 2z| < 1$
H: $|1 + z + z^2 + \frac{1}{24}z^3| < 1$

Question 6 Consider a general time-stepping scheme $\boldsymbol{y}_{i+1} = \mathcal{Q}(\boldsymbol{y}_i)$ applied to an initial-value problem $y' = f(y)$ with exact solution $y(t)$. For example forward-Euler corresponds to $\mathcal{Q}(y) = y + hf(y)$. Let the scheme predict a discrete solution $\boldsymbol{y}_i$ at time t_i. Which of the following statements are true?

(i) The *global-truncation error* at time t_i is $|y(t_i) - \boldsymbol{y}_i|$.

(ii) The *local-truncation error* at time t_i is $|y'(t_i) - \mathcal{Q}(y(t_i))|$.

(iii) If $y(t)$ is a degree-2 polynomial, and $\mathcal{Q}$ has local-truncation error proportional to h^3, then $y(t_i) = \boldsymbol{y}_i$ for any $\mathcal{Q}$.

A: Unanswered
B: None
C: (i)
D: (ii)
E: (i) and (ii)
F: (i) and (iii)
G: (ii) and (iii)
H: (i), (ii) and (iii)

Numerical optimization

Question 7 Consider the following function:

$$f(x, y) = -x^3 - y^2 + 3x - 3xy$$

Newton's method for *optimization* is applied and the following point extremal point $(x, y) = (2, -3)$ is found. What can be said about this point? It is:

A: Unanswered
B: A global maximum
C: A global minimum
D: A local maximum
E: A local minimum
F: A saddle-point
G: Not extremal

Question 8 A rectangular cold storage box with *square* base of edge length l meters, height h meters and perfectly insulated top has a total volume of $10\,\mathrm{m}^3$. All other sides are uninsulated. The goal is to find l such that heat loss is minimized.
Apply Newton's method for optimization to minimize a suitable objective function (assume heat loss is proportional to surface area). Perform the update in terms of l, starting with an initial estimate $l_0 = 1.0$. What is l_1 to two decimal places?

A: Unanswered	C: 1.46	E: 1.23	G: 2.46
B: 0.46	D: 1.19	F: 0.54	H: 1.54

Question 9 Apply 1 iteration of the steepest descent method to the quadratic form

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \cdot A \cdot \mathbf{x} + \mathbf{b}^T \cdot \mathbf{x} + c,$$

where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \quad c = 5.$$

with the initial condition $x_0 = (1, 1)$. What is steepest descent direction?

A: Unanswered	C: (1, 7)	E: (7, 1)	G: (3, −1)
B: (−1, −7)	D: (−7, −1)	F: (−3, 1)	H: (−1, 3)

Question 10 The steepest descent method is applied to the quadratic form

$$Q(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{x}^T \cdot A \cdot \boldsymbol{x} - \boldsymbol{b}^T \cdot \boldsymbol{x} + c,$$

where A, $\boldsymbol{b}$ and c, are matrix, vector and scalar constants. Under what condition on the matrix A does the steepest descent method converge to the exact minimum in 1 iteration, from *any* initial condition $\boldsymbol{x}_0$? [Hint: If the initial search line $x_0 + \alpha d_0$ includes the exact minimum of $Q(\boldsymbol{x})$, then the method will converge in 1 iteration.]

A: Unanswered	E: A is positive definite
B: A is a multiple of the identity matrix	F: A has only positive eigenvalues
C: A is diagonal	G: A is equal to $\boldsymbol{b}\boldsymbol{b}^T$
D: A is symmetric	H: It always converges in 1 iteration

END OF EXAM