Applied Numerical Analysis – AE2220-I – Resit

Modules 1–6 — Thursday, 27 June 2019, 13:30–16:30

Name: ____ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the **version number** of your quiz (see bottom right, 1-4) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **24 questions** and **8 pages** in total.

Module 1: Taylor, Root-finding, Floating-point

Question 1 Consider the positive floating point number system $z = s \times 10^e$ where the base is 10. A total of 8 decimal digits are used to store s and e. What is the largest number that can be represented using this number-system if a machine epsilon of at least 1×10^{-6} must be achieved?

A: Unanswered
B: 9.99999×10^8
C: 9.999999×10^8
D: 9.99999×10^9
E: 9.999999×10^9
F: 1.0×10^8
G: 1.0×10^9
H: 1.0×10^{10}

Question 2 Write $f(x) = \sin x$ as a truncated Taylor series expansion about $x = 0$ with two *non-zero* terms. What is approximately the size of the first non-zero term in the truncation error at $x = \frac{\pi}{2}$?

A: Unanswered
B: 0.008
C: 0.05
D: 0.06
E: 0.07
F: 0.08
G: 0.09
H: ∞

Question 3 What is the approximation of the root of the function $f(x) = e^x - 1$, if three steps of repeated bisection are applied on a starting interval $[x_1, x_2] = [-3, 2]$? [Hint: To apply the method quickly it may help to roughly plot the function and spot the root.]

A: Unanswered
B: -0.5
C: -0.2
D: -0.125
E: 0
F: 0.125
G: 0.15
H: 0.75

Question 4 A fixed-point iteration is applied to the function

$$f(x) := e^{a(x-1)} - x,$$

where $a \geq 0$ is a constant. This function has an exact root at $\tilde{x} = 1$ for all a. The specific iteration used is:

$$x_{n+1} := \varphi(x_n) = e^{a(x_n-1)},$$

and the initial guess is extremely close to one: $x_0 = 1 + \epsilon$, where $\epsilon > 0$ we can choose as small as we like. Under what condition on a does this iteration converge to 1?

A: Unanswered
B: $a < 2$
C: $a < 1$
D: $a = 0$
E: $a > 1$
F: $a > 2$
G: Always converges to 1.
H: Always diverges.

Module 2: Polynomial Interpolation and Regression

Question 5 Consider the function $f(x) = x^3$. Let $p_2(x)$ be a degree-2 polynomial which interpolates $f(x)$ at $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. We define the L_1-norm of the interpolation error on $[a, b]$ as:

$$\varepsilon = \int_a^b |p_2(x) - f(x)| dx.$$

What is the value of ε on the interval $[0, 1]$?

A: Unanswered
B: 0
C: $-\frac{1}{3}$
D: $\frac{1}{3}$
E: $-\frac{1}{4}$
F: $\frac{1}{4}$
G: $-\frac{8}{3}$
H: $\frac{8}{3}$

Question 6 By Cauchy's theorem the interpolation error for a degree n-polynomial is

$$R_n(f; x) := f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x),$$

for suitable $\xi(x)$ and $\omega(\cdot)$. Based on this formula, for the interpolation problem of Question 5, what is the maximum interpolation error of any point in $[0, 1]$?

A: Unanswered
B: $\sqrt{3}/9$
C: $2\sqrt{3}/9$
D: $3\sqrt{3}/9$
E: $4\sqrt{3}/9$
F: $2\sqrt{3}/3$
G: $\sqrt{3}$
H: $4\sqrt{3}/3$

Question 7 A data-set (x_i, f_i) containing $N + 1$ points with x_i all distinct, is approximated in 3 ways:

a) Polynomial interpolation, giving $p(x)$

b) Least-squares regression, giving $q(x)$

c) Regression, minimizing $\min \sum_{i=0}^{N} |f_i - r(x_i)|$, giving $r(x)$

In the two regression cases, $q(x)$ and $r(x)$ are degree M polynomials with $M < N$. Which of the following is true *in general*?

A: Unanswered
B: $p = q = r$
C: $p = q \neq r$
D: $p \neq q \neq r$
E: $p \neq q = r$

Question 8 Consider approximating a polynomial with a Fourier series. Using the basis functions

$$\varphi_0(x) = 1, \quad \varphi_1(x) = \cos x, \quad \varphi_2(x) = \sin x,$$

interpolate $f(x) = x^2$ using uniformly-spaced nodes on the interval $[0, \pi]$. What is the derivative of the interpolant at $x = 0$?

A: Unanswered
B: 0
C: $\pi/4$
D: $-\pi/4$
E: $\pi/2$
F: $-\pi/2$
G: $\pi^2/4$
H: $-\pi^2/4$

Question 9 An interpolating polynomial passes through the points

x	3	5	x_2
y	20	12	30

The polynomial is of the form:

$$p(x) = a_0 + a_1(x-3) + 2(x-3)(x-5).$$

What are the two possible values for x_2?

A: Unanswered
B: $x_2 = 0, 9$
C: $x_2 = 1, 7$
D: $x_2 = 2, 8$
E: $x_2 = 2, 6$
F: $x_2 = 3, -5$
G: $x_2 = 4, -6$

EXAM CONTINUES ON NEXT PAGE

Module 3: Advanced interpolation

Question 10 Consider the function $f(x,y) = 1/(x+y+1)$. Evaluate this function at the nodes given in the table, and linearly interpolate (triangular patch). What is the value of the interpolant at the point $(x,y) = (1/2, 1/2)$?

i	1	2	3
x_i	0	1	1
y_i	0	0	1

A: Unanswered
B: 1
C: $\frac{1}{2}$
D: $\frac{2}{3}$
E: $\frac{1}{4}$
F: $\frac{2}{5}$
G: $\frac{1}{6}$
H: $\frac{2}{7}$

Question 11 Which of the following functions is not a linear, quadratic or cubic spline?

A: Unanswered

B:

$$f(x) = \begin{cases} 2x^3 - \frac{9}{2}x^2 + 5x - \frac{3}{2}, & x \in [0,1] \\ 3x^3 - \frac{15}{2}x^2 + 8x - \frac{5}{2}, & x \in [1,2] \end{cases}$$

C:

$$f(x) = \begin{cases} 3x^3 - 8x^2 + 8x - 1, & x \in [0,1] \\ 2x^3 - 5x^2 + 3x + 2, & x \in [1,2] \end{cases}$$

D:

$$f(x) = \begin{cases} 2x - 1, & x \in [0,1] \\ 3x - 2, & x \in [1,2] \end{cases}$$

E:

$$f(x) = \begin{cases} 2x^2 - 2x + 1, & x \in [0,1] \\ 3x^2 - 4x + 2, & x \in [1,2] \end{cases}$$

Question 12 Given the function values of $f(x,y) = x^2 + 2xy + \sqrt{y} + 1$ at six nodes (the nodes are given in the table), we want to make an interpolation using the radial function $\varphi(r) = e^{-r^2}$. The interpolation condition leads to a linear system of dimension 6, which can be written as $\mathbf{A}\,\mathbf{a} = \mathbf{f}$, where $\mathbf{a}$ contains the unknown coefficients and $\mathbf{f}$ is the vector with given function values $\mathbf{f} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{pmatrix}^T$. Compute the element $A_{5,6}$ of the matrix $\mathbf{A}$.

i	1	2	3	4	5	6
x_i	1	1	2	2	3	3
y_i	1	3	3/2	4	2	5/2

A: Unanswered
B: 0.00
C: 0.12
D: 0.67
E: 0.78
F: 1.00
G: 1.18
H: 1.28

Module 4: Numerical differentiation and Integration

Question 13 Consider the one-sided difference formula for the 1st-derivative of $f(x)$:

$$f'(x) \approx \frac{1}{h}\left[3f(x) - 4f(x-h) + f(x-2h)\right].$$

What is the order of the truncation error of this approximation?

A: Unanswered
B: $\mathcal{O}(h)$
C: $\mathcal{O}(h^2)$
D: $\mathcal{O}(h^3)$
E: $\mathcal{O}(h^4)$
F: The approximation is exact.
G: The approximation is not consistent (error $\mathcal{O}(1)$).

Question 14 A high-order central-difference formula for the 1st-derivative of $f \in C^6([a,b])$ is:

$$f'(x_i) = \frac{f(x_i - 2h) - 8f(x_i - h) + 8f(x_i + h) - f(x_i + 2h)}{12h} + \epsilon$$

where ϵ is the truncation error. What is the value of the term in ϵ involving $f^{(5)}(x_i)$?

A: Unanswered
B: $-\frac{h^4 f^{(5)}(x_i)}{30}$
C: $-\frac{h^4 f^{(5)}(x_i)}{45}$
D: $-\frac{h^4 f^{(5)}(x_i)}{60}$
E: $-\frac{h^4 f^{(5)}(x_i)}{75}$
F: $-\frac{h^4 f^{(5)}(x_i)}{90}$

Question 15 What is the degree of precision of the quadrature formula

$$\int_0^1 f(x)\,dx \approx \frac{1}{2}\Big(f(-\sqrt{3}/3) + f(\sqrt{3}/3)\Big)?$$

A: Unanswered
B: 0
C: 1
D: 2
E: 3
F: 4
G: 5
H: 6

Question 16 **Fourier quadrature**. Until now we have considered quadrature rules that integrate polynomials of degree d exactly. Using the same principles derive a quadrature rule that integrates the functions $\phi = [1, \sin(x), \cos(x)]$ exactly on the interval $x \in [0, \pi]$. For nodes use $x = (0, \frac{\pi}{2}, \pi)$. What are the corresponding weights?

A: Unanswered
B: $w = (-1, 2+\pi, -1)$
C: $w = (\frac{\pi}{2}, 0, \frac{\pi}{2})$
D: $w = (\frac{\pi}{2}+1, -2, \frac{\pi}{2}+1)$
E: $w = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$
F: $w = (\frac{\pi}{3}-1, \frac{\pi}{3}+2, \frac{\pi}{3}-1)$
G: $w = (0, \pi, 0)$
H: $w = (\frac{\pi}{2}-1, 2, \frac{\pi}{2}-1)$

Module 5: Numerical solution of ODEs

Question 17 Consider a difference scheme $\boldsymbol{y}_{i+1} = \mathcal{D}(\boldsymbol{y}_i)$ applied to an initial-value problem with exact solution $y(t)$. Let the scheme predict a discrete solution $\boldsymbol{y}_i$ at time t_i. Which of the following statements are true?

(i) The *global error* at time t_{i+1} is $|y(t_{i+1}) - \boldsymbol{y}_{i+1}|$.

(ii) The *local error* at time t_{i+1} is $|\mathcal{D}(y(t_i)) - y(t_{i+1})|$.

(iii) If $y(t)$ is a polynomial of degree ≤ 1, and $\mathcal{D}$ is at least 1st-order accurate, then $y(t_i) = \boldsymbol{y}_i$ for all i.

A: Unanswered
B: None
C: (i)
D: (ii)
E: (i) and (ii)
F: (i) and (iii)
G: (ii) and (iii)
H: All

Question 18 Which of the following numerical schemes are *implicit*?

(i) $y_{n+1} = y_n + \Delta t f(y_{n+1})$

(ii) $y_{n+1} = y_n + \Delta t f(y_n)$

(iii) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(y_{n+1})\right)$

(iv) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(y_{n-1})\right)$

(v) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(\hat{y})\right), \quad \hat{y} = y_n + \Delta t f(y_n)$

A: Unanswered
B: (i), (iii)
C: (ii), (iv)
D: (ii), (iii), (iv)
E: (ii), (iv), (v)
F: (ii), (iii), (iv), (v)
G: (iii), (iv), (v)
H: (iv), (v)

Question 19 Consider the linear ODE

$$y'' + 2\lambda y' + \lambda^2 y = 0.$$

For what real values of λ is the solution y stable?

A: Unanswered
B: $\lambda < 1/\sqrt{2}$
C: $\lambda > 1/\sqrt{2}$
D: $\lambda > 1$
E: $\lambda < 1$
F: $\lambda < 0$
G: $\lambda > 0$

Question 20 The ODE $u' = -5u$ with initial condition $u(0) = u_0$ is solved numerically using (a) the forward-Euler method (explicit), and (b) the backward-Euler method (implicit). In both cases the stepsize $\Delta t = 0.3$. Of the two methods, which are stable?

A: Unanswered
B: Neither
C: (a) only
D: (b) only
E: Both

Module 6: Numerical optimization

Question 21 Consider the problem of finding the maximum of the function $f(x) = 2\sin x - \frac{x^2}{10}$ with an initial guess $x_0 = 5.0$. The generic update for Newtons method for optimization reads:

$$x_{n+1} = x_n - \zeta(x_n).$$

What is the expression for $\zeta(x)$ for the given function $f(x)$?

A: Unanswered
B: $x - \tan x$
C: $\tan x$
D: $\frac{10\cos x - x}{-1 - 10\sin x}$
E: $\frac{20\sin x - x^2}{20\cos x - 2x}$

Question 22 You are jogging on a path which can be described by a curve $y = \log x$ (where the natural logarithm is used). Your car is parked at the origin $(0, 0)$. You want to stop jogging at the point closest to your car. Use Newton's method to minimize the *squared* distance to the car, with an initial guess of $x_0 = 1$. What is the value of x_1?

A: Unanswered
B: 1
C: $\frac{1}{2}$
D: $\frac{1}{3}$
E: $\frac{1}{4}$
F: $\frac{1}{5}$
G: $\frac{1}{6}$
H: $\frac{1}{8}$

Question 23 Steepest descent for minimizing the function $f(\boldsymbol{x}) : \mathbb{R}^N \to \mathbb{R}$ in N-dimensions requires the gradient $\nabla f(\boldsymbol{x})$ on each iteration i. Assuming the function $\nabla f(\boldsymbol{x})$ is not known explicitly, it can be approximated at $\boldsymbol{x}_i \in \mathbb{R}^N$ by a difference rule (e.g. forward/backward/central differences). What is the *minimum* number of evaluations of $f(\cdot)$ required to approximate the gradient in this way?

A: Unanswered
B: 1
C: 2
D: $N - 1$
E: N
F: $N + 1$
G: $2N$
H: N^2

Question 24 Quadratic forms always have a single stationary point, which may be a minimum, maximum, or saddle point. Consider the quadratic form

$$Q(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \cdot A \cdot \boldsymbol{x} + \mathbf{b}^T \cdot \boldsymbol{x} + c.$$

Under what condition does this function have a **minimum**? [Hint: If you get stuck consider first 1d and then 2d examples.]

A: Unanswered
B: A is a multiple of the identity matrix
C: A is diagonal
D: A is symmetric
E: A is positive definite
F: $(-A)$ is positive definite
G: A is equal to $\mathbf{b}\mathbf{b}^T$
