Applied Numerical Analysis – AE2220-I – Quiz #3

Modules 5 and 6 – Thursday 4th April, 2019

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the **version number** of your quiz (see bottom right, 1-4) on the answer form.
- Use only **pencil** on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 standard questions** and **4 pages (2 sheets)** in total.

1 Numerical integration

Question 1 Consider the integral:

$$I = \int_0^{1/2} \sin(\pi x)\, dx$$

What is the Simpson's Rule approximation to this integral, using nodes at $x = (0, 1/4, 1/2)$?

A: Unanswered
B: 0.159
C: 0.318
D: 0.319
E: 0.321
F: 1.001
G: 1.000
H: 1.102

Question 2 What is the degree of precision (DoP) of the quadrature rule with two nodes:

$$\int_{-1}^{1} f(x)\, dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)?$$

A: Unanswered
B: 0
C: 1
D: 2
E: 3
F: 4
G: 5
H: 6

Question 3 Consider approximation of the two-dimensional integral:

$$I = \int_{-1}^{1} \int_{-1}^{1} f(x, y)\, dx\, dy.$$

Given a quadrature rule in 1d $Q_x[f]$, a 2d rule can be obtained by applying the 1d rule in each variable successively. I.e.

$$F(y) = Q_x[f(x, y)], \quad \text{and then} \quad I \simeq Q_y[F(y)],$$

whereby the first application of Q is to the variable x, and results in a function F of y only (x has been integrated out). Given the 1d-rule from Question 2, and the function $f(x, y) = x^2 + y^2$, what is the resulting approximation of I?

A: Unanswered
B: 1/3
C: 2/3
D: 4/3
E: 9/5
F: 8/3
G: 10/3
H: 16/3

2 Numerical solution of ODEs

Question 4 If the error introduced per step of an unspecified stable time integration scheme is $O(\Delta t^4)$ where Δt is the step-size, what can you say *in general* about the the error in the solution at a fixed time T?

A: Unanswered
B: $O(\Delta t^2)$
C: $O(\Delta t^3)$
D: $O(\Delta t^4)$
E: $O(\Delta t^5)$
F: $O(\Delta t^6)$
G: $O(\Delta t^7)$
H: $O(\Delta t^8)$

Question 5 Consider the ODE $u' = -cu$ with initial condition $u(0) = 1$ and constant $c > 0$. Using backward Euler:

$$u_{i+1} = u_i + \Delta t f(u_{i+1})$$

and $\Delta t = \frac{1}{2}$, what is the approximation of $u(4)$?

A: Unanswered
B: $(1 + c/2)^{-8}$
C: $(1 + c/2)^8$
D: $(1 - c/2)^{-8}$
E: $(1 - c/2)^8$
F: $(c/2)^{-8}$
G: $(c/2)^8$

Question 6 Consider a general time-stepping scheme $\boldsymbol{y}_{i+1} = \mathcal{Q}(\boldsymbol{y}_i)$ applied to an initial-value problem $y' = f(y)$ with exact solution $y(t)$. For example forward-Euler corresponds to $\mathcal{Q}(y) = y + hf(y)$. Let the scheme predict a discrete solution $\boldsymbol{y}_i$ at time t_i. Which of the following statements are true?

(i) The *global error* at time t_i is $|y(t_i) - \boldsymbol{y}_i|$.

(ii) The *local error* at time t_{i+1} is $|\mathcal{Q}(y(t_i)) - y(t_{i+1})|$.

(iii) If $y(t)$ is a degree-1 polynomial, and $\mathcal{Q}$ is 1st-order accurate, then $y(t_i) = \boldsymbol{y}_i$.

A: Unanswered
B: None
C: (i)
D: (ii)
E: (i) and (ii)
F: (i) and (iii)
G: (ii) and (iii)
H: (i), (ii) and (iii)

Question 7 Consider the following multi-stage time-stepping scheme:

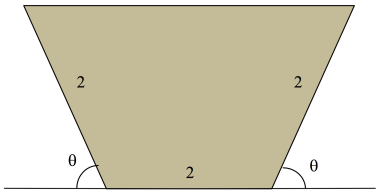
$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}h(k_1 + k_2), \\ k_1 &= f(y_n), \\ k_2 &= f(y_n + hk_1). \end{aligned}$$

This scheme is known as the 2nd-order Runge-Kutta method. Assume for simplicity that $f(y) = cy$, with c a constant. Let $z = ch$. For what values of z is the scheme stable?

A: Unanswered
B: $|1 + z| < 1$
C: $|1 - z| < 1$
D: $|1 + z + \frac{1}{2}z^2| < 1$
E: $|1 + z + \frac{1}{2}z^2 + \frac{1}{12}z^3| < 1$
F: $|1 + z + z^2| < 1$
G: $|1 + 2z| < 1$
H: $|1 + z + z^2 + \frac{1}{24}z^3| < 1$

3 Numerical optimization

Question 8 Consider the gutter-shape plotted below with base- and edges of length 2. Goal is to maximize the cross-sectional area of the gutter by varying the angle θ. Golden-section search is applied to solve this problem using an initial interval $I_0 = [0, \frac{\pi}{2}]$. Apply one iteration of this method to obtain I_1. What is the midpoint of I_1? [Note: In G-S search on the interval $[0, 1]$, the interior sample points are at $\frac{1}{\phi}$ and $1 - \frac{1}{\phi}$ with $\phi = \frac{1+\sqrt{5}}{2}$.]



A: Unanswered
B: $\frac{\pi}{2}$
C: $\frac{7\pi}{9}$
D: $\frac{\pi^2}{2}$
E: 1.085
F: 1.125

Question 9 We want to approximate π. We know that

$$\pi = \text{argmin}_{1 \leq x \leq 5} [\cos x] .$$

Apply 1 iteration of golden-section search. What is the *midpoint* of the interval after this single iteration?

A: Unanswered
B: 3.142
C: 3.564
D: 3.664
E: 3.764
F: 3.864
G: 3.964

Question 10 A rectangular cold storage box with *square* base of edge length l meters, height h meters and perfectly insulated top has a total volume of $10\,\text{m}^3$. All other sides are uninsulated. The goal is to find l such that heat loss is minimized.
Apply Newton's method for optimization to minimize a suitable objective function (assume heat loss is proportional to surface area). Perform the update in terms of l, starting with an initial estimate $l_0 = 1.0$. What is l_1 to two decimal places?

A: Unanswered
B: 0.46
C: 1.46
D: 1.19
E: 1.23
F: 0.54
G: 2.46
H: 1.54

END OF EXAM