Applied Numerical Analysis – AE2220-I – Quiz #2

Modules 3 and 4 – Friday 15th March, 2019

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the **version number** of your quiz (see bottom right, 1-4) on the answer form.
- Use only **pencil** on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 standard questions** and **4 pages (2 sheets)** in total.
- Question 11 is a bonus question, for extra points only.

1 Multi-dimensional interpolation

Question 1 We wish to interpolate the function of M variables $f(x_1, x_2, \ldots, x_M)$, with a tensor-product polynomial basis. We know *a priori* that the first P variables are the most important. Therefore to build the tensor-product basis, we use degree 5 polynomial approximations (1d) in each of the variables $x_1, \ldots, x_P$ individually, and degree 2 polynomial approximations in each of the variables $x_{P+1}, \ldots, x_M$ individually. How many basis functions are in the tensor-product basis in total?

A: Unanswered
B: $P^6(M-P)^3$
C: $P^5(M-P)^2$
D: $5^P 2^{M-P}$
E: $6^P 3^{M-P}$
F: 6^M
G: 5^P
H: 6^P

Question 2 The function $f(x, y)$ is interpolated at the points of the unit square $x_0 = (-1, -1)$, $x_1 = (1, -1)$, $x_2 = (1, 1)$ and $x_3 = (-1, 1)$, in two ways:

- With a single rectangular patch on (x_0, x_1, x_2, x_3), giving $p_4(x, y)$
- With two triangular patches, on (x_0, x_1, x_2) and (x_2, x_3, x_0) seperately, giving $p_3(x, y)$

Which of the following statements are true, for $x, y \in [-1, 1]$ and any f? [Hint: The basis for $p_4(x, y)$ is $(1, x, y, xy)$, the basis for $p_3(x, y)$ is $(1, x, y)$.]

I. $p_4(-1, y) = p_3(-1, y)$

II. $p_4(x, y) = p_3(x, y)$

III. $p_4(x, 1) = p_3(x, 1)$

A: Unanswered
B: All
C: I
D: II
E: III
F: I,III
G: I,II
H: None

Question 3 Radial-basis interpolation is used to approximate $f(x) = x$ in 1d using the nodes $x_0 = -1$, $x_1 = 1$, $x_2 = 2$, and the basis function $\phi(r) = \exp(-r^2)$. What is the value of the interpolant at $x = 100$, accurate to 2 significant figures?

A: Unanswered
B: 0.0
C: 1.0
D: 10
E: 50
F: 100

Question 4 We create an interpolant $s(x)$ for the function $f(x) = 1 - \frac{1}{2}x^2$ using radial basis function interpolation. We use data points at $x = 0, 1, 2$ and the following radial basis function:

$$\phi(r) = \begin{cases} 1 - \frac{r^2}{\theta^2} & \text{for } |r| \le \theta \\ 0 & \text{for } |r| > \theta \end{cases}$$

The radial distance between a point x_i and a point x_j is defined as $r = |x_i - x_j|$, and $\theta > 0$ is a tuning parameter. Which of the following statements is true?

I. $s(x) = f(x)$ for all $x \in [-\theta, \theta]$ when $\theta = \sqrt{2}$

II. When $0 < \theta \le 1$, the system matrix A is the identity matrix

III. The 2nd-derivative of $s(x)$ is constant

A: Unanswered
B: All
C: I
D: II
E: I,II
F: I,III
G: II,III
H: None

2 Spline interpolation

Question 5 Consider a generalized spline interpolant $s(x)$ on the interval $[0, 1]$. The interval is divided into sub-intervals with $N + 1$ nodes:

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$$

at which sample data are given. Assume that the spline consists of polynomials of degree d on each interval, and that the spline is required to be $d - 1$-times continuously differentiable on the entire interval.

We impose 2 constraints at the boundaries: $f''(0) = f''(1) = 0$. What is the degree d of the polynomials needed to always be able to construct such a spline?

A: Unanswered	C: 4	E: 6	G: 8
B: 3	D: 5	F: 7	H: 9

3 Regression

Question 6 Consider i) interpolation and ii) regression of a two-dimensional function $f(\mathbf{x})$ where $f : \mathbb{R}^2 \to \mathbb{R}$. Assume that the approximating function $p(\mathbf{x})$ in both cases is a linear combination of N distinct basis functions $\phi_i(\mathbf{x})$, and that f is sampled at M distinct locations. The interpolation conditions for i) lead to a system of linear equations, with system matrix A.

Consider the existence and uniqueness of solutions to these two problems. What condition is required on i) and ii) respectively, to guarantee unique solutions for each problem?

A: Unanswered
B: i) When $M = N$, and ii) when $M \geq N$.
C: i) When $\det(A) \geq 0$, and ii) $\det(A^T A) \geq 0$.
D: i) When $\det(A) \neq 0$, and ii) $\det(A^T A) \neq 0$.
E: i) Always for polynomial ϕ_i, and ii) polynomial ϕ_i.
F: i) Always for radial-basis functions ϕ_i, and ii) polynomial ϕ_i.
G: i) A invertable, and ii) A symmetric.

Question 7 Steady laminar flow through a channel with parallel walls has a *parabolic* velocity profile (“Poiseuille” flow). In a channel of height 2 on the interval $y \in [-1, 1]$ we have two *approximate* measurements of the velocity, at $y_0 = -\frac{1}{2}$ and $y_1 = \frac{1}{2}$ with values $v_0 = 7.3$ and $v_1 = 8.7$. In addition we know that the velocity at both channel walls is *exactly* zero (by the no-slip boundary condition), i.e. $v(-1) = v(1) = 0$. Derive and solve a mixed interpolation/regression problem to approximate $v(y)$, using a quadratic polynomial. What is the maximum velocity in the approximation of v?

A: Unanswered	C: 26/3	E: 36/5	G: 38/5
B: 25/3	D: 32/3	F: 37/5	H: ∞

4 Numerical differentiation

Question 8 One way of approximating derivatives is by rearranging a Taylor series. The Taylor approximation of $f(x)$ about x_0 is:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \mathcal{O}(h^3),$$

where $h := x - x_0$. Assuming the value of f is known only at x_0 and x_1, rearrange Taylor to get an approximate expression for $f'(x_0)$. What is the magnitude of the *error* in this approximation?

A: Unanswered
B: $\mathcal{O}(1)$
C: $\mathcal{O}(x_1 - x_0)$
D: $\mathcal{O}(x_1 - x_0)^2$
E: $\mathcal{O}(x_1 - x_0)^3$
F: $\mathcal{O}(x_1 - x_0)^4$
G: $\mathcal{O}(x_1 - x_0)^5$

Question 9 Assume you have obtained measurements of the lift coefficient of an aerofoil $C_L(\alpha)$ at 3 different values of the angle of attack α: $C_L(-2) = -0.32$, $C_L(0) = 0$, $C_L(2) = 0.4$. Which of the following techniques for estimating the derivative $\frac{\mathrm{d}C_L}{\mathrm{d}\alpha}$ at $\alpha = 0$ produces the same value?

I. Central difference formula

II. Derivative of a least-squares linear regression

III. Derivative of a quadratic polynomial interpolant

A: Unanswered
B: I, II and III
C: I, II
D: II, III
E: I, III
F: All give distinct answers

Question 10 Consider this proposal for a finite difference scheme:

$$f'(x) = \frac{2f(x + 2h) - f(x + h) - f(x)}{6h} + \epsilon.$$

How does the error ϵ behave in terms of h?

A: Unanswered
B: $\mathcal{O}(1)$ - inconsistent rule
C: $\mathcal{O}(h)$
D: $\mathcal{O}(h^{1.5})$
E: $\mathcal{O}(h^2)$
F: $\mathcal{O}(h^3)$

5 Bonus points

Question 11 Today, 15th March 2019, there is a national strike in progress in the Netherlands. In which sector is this strike?

A: Unanswered
B: Public transport
C: Air-traffic control
D: Police
E: Health care
F: Education
G: Rubbish collection
H: Dentistry

END OF EXAM