Applied Numerical Analysis – AE2220-I – Quiz #1

Modules 1 and 2 – Monday 4th March, 2019

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the **version number** of your quiz (see bottom right, 1-4) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **5 pages (3 sheets)** in total.

1 Computer representation of numbers

Question 1 Consider the positive floating point number system $s \times b^e$, where $b = 10$, s is a 6-digit significant $1.00000 \leq s \leq 9.99999$ and $-7 \leq e \leq 8$. What is the smallest *integer* strictly greater than 0, which is not representable by this system?

A: Unanswered	C: 999991	E: 1000000	G: 10000001
B: 100001	D: 999999	F: 1000001	H: 99999001

Question 2 The decimal (i.e. base $b = 10$) number system of Question 1 is to be implemented on a binary computer (i.e. base $b = 2$). How many bits are needed to be able to represent any number in the system? [Hint: Make a counting argument.]

A: Unanswered	C: 20 bits	E: 28 bits	G: 36 bits
B: 16 bits	D: 24 bits	F: 32 bits	H: 64 bits

2 Taylor series approximation

Question 3 What is the Taylor expansion of $\cosh(x)$ about $x = 0$? Note: $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.

A: Unanswered	E: $\sum_{i=0}^{\infty}(-x)^i/i!$
B: $\sum_{i=0}^{\infty} x^i/i!$	F: $\sum_{i=0}^{\infty}(-1)^i x^{2i}/(2i)!$
C: $\sum_{i=0}^{\infty} x^{2i}/(2i)!$	G: $\sum_{i=0}^{\infty}(-1)^i x^{2i+1}/(2i+1)!$
D: $\sum_{i=0}^{\infty} x^{2i+1}/(2i+1)!$	H: 1

Question 4 Using a Taylor expansion about zero we can write

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

Using the Lagrange remainder for a truncated Taylor series, express the infinite sum

$$z = \sum_{i=10}^{\infty} \frac{2^i}{i!}$$

as a single term involving the unknown $\xi \in [0, 2]$. [Note: By this method one can deduce that the Taylor series of e^x converges for all $x \in \mathbb{R}$ – a property which is not guaranteed for arbitrary functions.]

A: Unanswered	C: $z = e^\xi/11!$	E: $z = e^\xi 2^{11}/11!$	G: $z = \xi^{11}/11!$
B: $z = e^\xi/10!$	D: $z = e^\xi 2^{10}/10!$	F: $z = \xi^{10}/10!$	

3 Root-finding methods

Question 5 The recursive bisection method is guaranteed to converge to a root for continuous functions ($C^0([a,b])$) only, however it *can be applied* whenever the function has a different sign at the left and right side of the interval. Consider a large (but finite) number of recursive bisection iterations applied to the function $f(x) = \frac{1}{x-\frac{1}{9}}$ on the initial interval $[a_0, b_0] = [0, 1]$. What will happen?

A: Unanswered
B: The method can not be applied.
C: The method eventually samples f *at* the singularity and fails.
D: The interval halves while containing the singularity at every step.
E: The left interval is always chosen, and we converge to 0.
F: The right interval is always chosen, and we converge to 1.

Question 6 The fixed-point iteration

$$x_{i+1} = \frac{1}{2}(x_i + \frac{z}{x_i}),$$

(known in antiquity as Heron's method) is a method for computing the square-root of z. Assume a starting guess x_0 is chosen such that the method converges. What value does the *convergence rate* approach as the error approaches zero? [Note: The convergence rate is the factor by which the error reduces at each iteration.]

A: Unanswered
B: $\sqrt{z}$
C: z
D: $1/z^2$
E: $1/z$
F: $1/\sqrt{z}$
G: $1/2$
H: 0

Question 7 Newton's method is based on a linear approximation of f at the iterate x_i. Phaedrus conceives of an "improved" method based on a *quadratic* approximation of f at x_i. Assuming that the initial guess is $x_0 = 0$, what is the expression for x_1 with Phaedrus's method? [Note: Below f, f' and f'' are short for $f(x_0)$, $f'(x_0)$, and $f''(x_0)$ respectively.] [Hint: Use a Taylor expansion of f to find the approximating curve.]

A: Unanswered
B: $x_1 = -f/f'$
C: $x_1 = -f/f''$
D: $x_1 = -f'/f''$
E: $x_1 = (-f' \pm \sqrt{f'^2 - 2ff''})/f''$
F: $x_1 = (-f' \pm \sqrt{f'^2 - 4ff''})/2f''$
G: $x_1 = \pm f''f/f'^2$
H: $x_1 = \pm f'f/f''^2$

4 Polynomial interpolation

Question 8 Consider a polynomial interpolant $p_N(x)$ of the function

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

on the interval $x \in [-1, 1]$, using a Chebychev grid with N nodes where N is even (avoiding a node at 0). Which of the following statements are true?

1. The function is discontinuous, so a polynomial interpolant doesn't exist.
2. By the Weierstrass theorem there exists a polynomial such that the interpolation error is less than $\epsilon > 0$ for any ϵ.
3. By Cauchy's interpolation error theorem the interpolation error is

$$f(x) - p_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}\,\omega_{N+1}(x),$$

   where $\omega_{N+1}(x)$ is the nodal polynomial.
4. The maximum interpolation error approaches 0 as $N \to \infty$.

A: Unanswered
B: All
C: 1
D: 2
E: 2,3
F: 2,4
G: 2,3,4
H: None

Question 9 Suppose that to interpolate a function $f(x)$, the *Newton basis*

$$\pi_0(x) = 1$$
$$\pi_1(x) = (x - x_0)$$
$$\cdots$$
$$\pi_N(x) = \prod_{i=0}^{N-1} (x - x_i)$$

is chosen as a basis for $\mathbb{P}_N$, the space of polynomials of degree N. Given that the interpolation nodes are $(x_0, x_1, \ldots, x_N)$, what is the value of the determinant of the interpolation matrix ($\det(A)$)? [Note: The answer can be compared with the determinant of the Vandermonde matrix.]

A: Unanswered
B: 1
C: $\sum_{i=1}^{N} \prod_{j=0}^{i} (x_i - x_j)$
D: $\prod_{i=1}^{N} \prod_{j=0, j\neq i}^{N} (x_i - x_j)$
E: $\prod_{i=1}^{N} \prod_{j=0}^{N} (x_i - x_j)$
F: $\prod_{i=1}^{N} \prod_{j=0}^{i-1} (x_i - x_j)$
G: $\sum_{i=1}^{N} \prod_{j=0}^{N} (x_i - x_j)$

Question 10 Consider the function $f(x) = \sin(x)$. Assume we know the value of the function only at 3 equidistant points: $(x_0, x_1, x_2) = (-h, 0, h)$. We interpolate this function using a second degree polynomial. We know an upper bound for the interpolation error for a degree N polynomial from Cauchy's theorem:

$$|f(x) - p_N(x)| \leq \max_{\xi \in [a,b]} |f^{N+1}(\xi)| \frac{|\omega_{N+1}(x)|}{(N+1)!},$$

where ω is the nodal polynomial. Based on this result, what is a bound on the error in the specific case above?

A: Unanswered
B: h^2
C: $h^2/(4\sqrt{2})$
D: $h^3/18$
E: $h^3/(18\sqrt{3})$
F: $h^3/(9\sqrt{3})$
G: $h^4/(27\sqrt{3})$
H: $h^4/(81\sqrt{3})$

END OF EXAM