Applied Numerical Analysis – Resit

3 hours — Modules 1–6

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **24 questions** and **8 pages** in total.

Module 1: Taylor, Root-finding, Floating-point

Question 1 Consider the positive floating point number system $z = s \times 10^e$ where the base is 10. A total of 8 decimal digits are used to store s and e. What is the largest number that can be represented using this number-system if a machine epsilon of at least 1×10^{-5} must be achieved?

A: Unanswered
B: 9.99999×10^4
C: 9.99999×10^5
D: 9.99999×10^6
E: 9.99999×10^7
F: 9.99999×10^8
G: 9.99999×10^9
H: 9.99999×10^{99}

Question 2 The infinite sum

$$s = \sum_{n=0}^{\infty} \frac{2^n}{n!}$$

is the result of evaluating a Taylor-series about $x_0 = 0$ of one of the basic functions $\cos x$, $\sin x$, e^x or $\log(1 + x)$ at a particular value of x. Given this information, what is the value of s?

A: Unanswered
B: 0
C: $\cos 2$
D: $\cos 3$
E: e^2
F: e^3
G: $\log(2)$
H: $\log(3)$

Question 3 The small-angle approximation for trigonometric functions is based on a Taylor expansion about $x = 0$, up to quadratic terms. For which angle does approximation of $\sin(x)$ have a relative error exceeding approximately 1.0%? [Hints: Relative error is defined as $\frac{f_{true} - f_{approx}}{f_{true}} \times 100\%$. Rather than solving the resulting equation, each of the given options can be checked.]

A: Unanswered
B: $7\pi/180$
C: $14\pi/180$
D: $21\pi/180$
E: $\pi/2$
F: π

Question 4 We propose a variant of recursive bisection, where each interval $[a_i, b_i]$ is not divided in the center, but into two unequal pieces. The left piece is $\frac{1}{5}$ the interval width, and the right piece is the remaining $\frac{4}{5}$. I.e. $x_i = \frac{4a_i + b_i}{5}$. Otherwise the algorithm is the same.
The rate of convergence of the standard algorithm is $\frac{1}{2}$. Assuming that the root can be anywhere within the interval, what is the *average* rate of convergence of the modified method? [Hint: take into account the probability of the root landing in each sub-interval.]

A: Unanswered
B: $\frac{1}{5}$
C: $\frac{4}{5}$
D: $\frac{1}{2}$
E: $\frac{13}{25}$
F: $\frac{17}{25}$

Module 2: Polynomial Interpolation and Regression

Question 5 We wish to perform regression with the approximant $\phi(x) = a_0 + a_1 \ln(a_2 x)$. In order to fit the pairs of points (x_i, f_i), $i \in \{0, N\}$ we minimise the sum of squared-residuals,

$$\psi = \sum_{i=0}^{N} (\phi(x_i) - f_i)^2,$$

by solving $\frac{\partial \psi}{\partial a} = 0$ for a_0,a_1,a_2. Consider the statements:

1. The system of equations is linear.
2. The system of equations can be solved using the recursive-bisection method.
3. The system of equations can be solved using a fixed-point iteration.
4. The system of equations can be solved using the Newton method.

Which of the above are true?

A: Unanswered
B: 1
C: 2
D: 2, 3
E: 3, 4
F: 1, 2, 3
G: 1, 2, 4

Question 6 Interpolate a function $f(x)$ with polynomials. You choose to use the monomial, the Newton and the Lagrange basis on the same grid (nodes) to get the interpolants $p_1(x)$, $p_2(x)$ and $p_3(x)$ respectively. What can be said about $p_1(x)$, $p_2(x)$ and $p_3(x)$?

A: Unanswered
B: They are *always* different
C: They *can* be different
D: $p_1(x) = p_2(x) = p_3(x)$
E: $p_2(x) \neq p_3(x)$
F: $p_2(x) \neq p_1(x)$
G: $p_1(x) \neq p_3(x)$
H: None of the above

Question 7 Consider the interpolant

$$\phi(x) = \sum_{k=0}^{n} a_k \cos(kx) \sin^2((k+1)x)$$

and the nodes $\mathbf{x} = (0, \pi/2, \pi)$. For a suitable choice of n the interpolation problem can be written $\mathbf{Aa} = \mathbf{f}$, where $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{R}^3$ are the interpolation coefficients and $\mathbf{f} \in \mathbb{R}^3$ are samples of a function $f(x)$ at $\mathbf{x}$. What is the value of the element $\mathbf{A}_{22}$ of the interpolation matrix?

A: Unanswered
B: 0
C: $1/\sqrt{2}$
D: 1
E: $\pi/2$
F: π

Question 8 Suppose a table is to be prepared for the function $f(x) = \sqrt{x}$ on the interval $[a, b] = [1, 2]$ with equal spacing h. Determine h, such that the interpolation with a polynomial of degree 2 will give an accuracy of $\varepsilon = 5 \times 10^{-8}$.
Hint: For equidistant nodes and polynomial degree n the error satisfies:

$$E_n \leq \max_{\xi\in[a,b]} |f^{(n+1)}(\xi)| \, \frac{|h^{n+1}|}{4(n+1)}.$$

A: Unanswered
B: 0.0117
C: 0.0015
D: 0.131
E: 0.02
F: 0.5
G: 0.032
H: 0.148

Module 3: Advanced interpolation

Question 9 Consider the function $f(x, y) = 3/(x + y + 1)$. Evaluate this function at the nodes given in the table, and linearly interpolate (triangular patch). What is the value of the interpolant at the point $(x, y) = (1/2, 1/2)$?

i	1	2	3
x_i	0	1	1
y_i	0	0	1

A: Unanswered
B: 0
C: $\frac{1}{2}$
D: 1
E: $\frac{3}{2}$
F: 2
G: $\frac{5}{2}$
H: 3

Question 10 A cubic spline interpolating $f(x)$ at $x_0 < x_1 < ... < x_N$ is

$$\phi(x) = \begin{cases} S_0(x) & \text{in } x_0 \leq x \leq x_1 \\ S_1(x) & \text{in } x_1 \leq x \leq x_2 \\ \cdots & \cdots \\ S_{N-1}(x) & \text{in } x_{N-1} \leq x \leq x_N \end{cases}$$

where S_i are cubic polynomials

$$S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i.$$

To be a spline $\phi(x)$ must interpolate $f(x_i)$ and both $\phi'(x)$ and $\phi''(x)$ must be continuous. Under these constraints the system has multiple solutions for the coefficients $a_i, ..., d_i$, i.e. the interpolation is not unique.
Which one of the following additional constraints does not result in a unique spline?

A: Unanswered
B: $S_0''(x_0) = S_N''(x_N) = 0$
C: $S_0'(x_0) = f'(x_0)$; $S_N'(x_N) = f'(x_N)$
D: $S_0''(x_0) = f''(x_0)$; $S_N''(x_N) = f''(x_N)$
E: $S_1'''(x_1) = S_{N-1}'''(x_{N-1}) = 0$
F: $S_0'''(x_1) = S_1'''(x_1)$; $S_{N-2}'''(x_{N-1}) = S_{N-1}'''(x_{N-1})$
G: All above result in a unique spline

Question 11 Which of the following functions is not a linear, quadratic or cubic spline?

A: Unanswered
B:

$$f(x) = \begin{cases} 2x^3 - \frac{9}{2}x^2 + 5x - \frac{3}{2}, & x \in [0,1] \\ 3x^3 - \frac{15}{2}x^2 + 8x - \frac{5}{2}, & x \in [1,2] \end{cases}$$

C:

$$f(x) = \begin{cases} 3x^3 - 8x^2 + 8x - 1, & x \in [0,1] \\ 2x^3 - 5x^2 + 3x + 2, & x \in [1,2] \end{cases}$$

D:

$$f(x) = \begin{cases} 2x - 1, & x \in [0,1] \\ 3x - 2, & x \in [1,2] \end{cases}$$

E:

$$f(x) = \begin{cases} 2x^2 - 2x + 1, & x \in [0,1] \\ 3x^2 - 4x + 2, & x \in [1,2] \end{cases}$$

Question 12 We interpolate $f(x, y)$ at $N + 1$ points (x_i, y_i), all lying on a circle of radius R, using radial basis-function interpolation, with the radial function $\phi(r) = \exp(-r^2)$. The resulting interpolant takes the value M at the center of the circle. What is the sum of the unknown coefficients of the radial-basis interpolant, $\sum_{i=0}^{N} a_i$?

A: Unanswered
B: N
C: $N\exp(R^2)$
D: $N\exp(-R^2)$
E: $M\exp(R^2)$
F: $M\exp(-R^2)$
G: $\exp(-NR^2)$
H: Insufficient information

Module 4: Numerical differentiation and Integration

Question 13 Consider an infinitely differentiable function $f(x)$ and the integral

$$I = \int_a^b f(x)dx.$$

Approximate I numerically as follows: (a) divide the interval $[a, b]$ into N equally sized intervals of width h, (b) approximate the integral on each interval by the mid-point rule (i.e. one point at the center of the interval).
Assume h is already small. When h is divided by 2, by what factor does the error in the approximation of I decrease?

A: Unanswered
B: 1
C: 2
D: 4
E: 8
F: 16

Question 14 Consider the following integral:

$$I = \int_0^1 \sin(\pi x)\,\mathrm{d}x$$

approximated by the trapezoidal rule. How many sub-intervals does the interval $[0, 1]$ need to be divided into to evaluate I with an error $< 1 \times 10^{-2}$?

A: Unanswered
B: 1
C: 2
D: 4
E: 8

Question 15 Given the forward-difference formula for the first derivative:

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(x)}{2!}\,h - \frac{f^{(3)}(x)}{3!}\,h^2 - \frac{f^{(4)}(x)}{4!}\,h^3 - \ldots$$

Apply Richardson extrapolation to derive a scheme of one order higher. What is the coefficient of $f(x + \frac{h}{2})$ in this new scheme?

A: Unanswered
B: 0
C: $1/h$
D: $2/h$
E: $2/h^2$
F: $4/h$
G: $4/h^2$

Question 16 **Hyperbolic quadrature**. Until now we have considered quadrature rules that integrate polynomials of degree d exactly. Using the same principles derive a quadrature rule that integrates the functions $\phi = [1, \sinh(x), \cosh(x)]$ exactly on the interval $x \in [-1, 1]$. For nodes use $x = (-1, 0, 1)$. What is the weight of the node at $x = 1$? [Note: $\cosh(x) = (e^x + e^{-x})/2$, $\sinh(x) = (e^x - e^{-x})/2$.]

A: Unanswered
B: $\frac{\sinh(1)}{\cosh(1)}$
C: $\frac{\cosh(1)}{\sinh(1)}$
D: $\frac{\sinh(1)+1}{\cosh(1)+1}$
E: $\frac{\cosh(1)+1}{\sinh(1)+1}$
F: $\frac{\sinh(1)-1}{\cosh(1)-1}$
G: $\frac{\cosh(1)-1}{\sinh(1)-1}$
H: No such rule exists.

Module 5: Numerical solution of ODEs

Question 17 Consider the system of two non-linear first-order ODEs

$$z_1' = z_2 \qquad z_2' = -\frac{g}{l}\sin(z_1)$$

describing an ideal pendulum, in which z_1 is the phase angle and z_2 is its time derivative. Use forward-Euler and the initial condition $(z_1^{(0)}, z_2^{(0)}) = (\frac{\pi}{2}, 0)$ (superscripts indicate the timestep). Assume $\frac{g}{l} = 10$. What is the value of $(z_1^{(1)}, z_2^{(1)})$ if $\Delta t = 0.001$?

A: Unanswered
B: $(\pi/2, -0.01)$
C: $(\pi/2, 0.01)$
D: $(-\pi/2, -0.01)$
E: $(-\pi/2, 0.01)$
F: $(0, -\pi/2)$
G: $(0, \pi/2)$

Question 18 Which of the following statements, pertaining to the linear ODE $\boldsymbol{u}' = A\boldsymbol{u}$, with A a matrix, are true?

(i) The ODE is (strictly) stable if solutions $u_1(t)$ and $u_2(t)$ with slightly different initial conditions have the property:
$$|u_1 - u_2| \to 0, \quad \text{as} \quad t \to \infty.$$

(ii) The ODE is stable if every eigenvalue of the matrix A has non-positive real part.

(iii) The ODE is stable if complex eigenvalues of A have equal real and imaginary parts.

(iv) The forward-Euler method (explicit) applied to the ODE is always stable.

A: Unanswered
B: None
C: (i), (ii)
D: (i), (ii), (iii)
E: (i), (ii), (iii), (iv)
F: (ii), (iii)
G: (ii), (iii), (iv)
H: All

Question 19 Consider a difference scheme $\boldsymbol{y}_{i+1} = \mathcal{D}(\boldsymbol{y}_i)$ applied to an initial-value problem with exact solution $y(t)$. Let the scheme predict a discrete solution $\boldsymbol{y}_i$ at time t_i. Which of the following statements are true?

(i) The *global error* at time t_i is $|y(t_i) - \boldsymbol{y}_i|$.

(ii) The *local error* at time t_i is $|\mathcal{D}(y(t_i)) - y(t_{i+1})|$.

(iii) If $y(t)$ is a polynomial of degree ≤ 1, and $\mathcal{D}$ is 1st-order accurate, then $y(t_i) = \boldsymbol{y}_i$.

A: Unanswered
B: None
C: (i)
D: (ii)
E: (i) and (ii)
F: (i) and (iii)
G: (ii) and (iii)
H: (i), (ii) and (iii)

Question 20 Consider the scheme:

$$u_{i+1} = u_i + \frac{3}{2}\Delta t f(u_i) + \frac{1}{2}\Delta t f(u_{i-1}),$$

where u_i and u_{i-1} are known, and u_{i+1} is unknown. What is the local-truncation error of this scheme? [Hint: Taylor expand u_{i-1}, then $f(u_{i-1})$.]

A: Unanswered
B: Δt^0
C: Δt^1
D: Δt^2
E: Δt^3

Module 6: Numerical optimization

Question 21 Steepest descent for minimizing $f(x, y)$ in 2-dimensions requires the gradient: $\nabla f(x_i, y_i)$ on iteration i. Assuming the function $\nabla f(x, y)$ is not known, it can be approximated at (x_i, y_i) by a difference rule (e.g. forward/backward/central differences). What is the *minimum* number of samples of $f(\cdot)$ required to approximate the gradient at (x_i, y_i) in this way?

A: Unanswered
B: 1
C: 2
D: 3
E: 4
F: 5

Question 22 You are jogging on a path which can be described by a curve $y = \log x$ (where the natural logarithm is used). Your car is parked at the origin $(0, 0)$. You want to stop jogging at the point closest to your car. Use Newton's method to minimize the *squared* distance to the car, with an initial guess of $x_0 = 1$. What is the value of x_1?

A: Unanswered
B: $\frac{1}{8}$
C: $\frac{1}{6}$
D: $\frac{1}{5}$
E: $\frac{1}{4}$
F: $\frac{1}{3}$
G: $\frac{1}{2}$
H: 1

Question 23 Consider the problem of finding the maximum of the function $f(x) = 2\sin x - \frac{x^2}{10}$ with an initial guess $x_0 = 2.5$. The generic update for Newtons method for optimization reads:

$$x_{n+1} = x_n - \zeta(x_n).$$

What is the expression for $\zeta(x)$ for the given function $f(x)$?

A: Unanswered
B: $x - \tan x$
C: $\tan x$
D: $\frac{10\cos x - x}{-1-10\sin x}$
E: $\frac{20\sin x - x^2}{20\cos x - 2x}$

Question 24 We want to approximate π. We know that

$$\pi = \arg\min_{1\leq x\leq 5} [\cos x] .$$

Apply 1 iteration of golden-section search. What is the midpoint of the interval after this single iteration?

A: Unanswered
B: 3.142
C: 3.264
D: 3.364
E: 3.464
F: 3.564
G: 3.664
H: 3.764
