Applied Numerical Analysis – Quiz #3
Modules 4 (integration), 5 and 6

Name: ___________________________________________ Student number: __________

DO NOT OPEN UNTIL ASKED

Instructions:

• Make sure you have a machine-readable answer form.
• Write your name and student number in the spaces above, and on the answer form.
• Fill in the answer form neatly to avoid risk of incorrect marking.
• Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
• Use only pencil on the answer form, and correct with a rubber.
• This quiz requires a calculator.
• Each question has exactly one correct answer.
• Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
• This quiz has 10 questions and 4 pages in total.
1 Numerical integration

Question 1 Expressions for the errors of a single trapezoidal and Simpson’s rule on an interval \([a, b]\) are:

\[ E_{\text{trap}}(f) = \frac{(b - a)^3}{12} f''(\xi), \quad E_{\text{simp}}(f) = \frac{(b - a)^5}{2880} f''''(\xi), \]

for some \(\xi \in [a, b]\). Consider now a composite trapezoidal-rule and a composite Simpson’s-rule, both used to approximate \(\int_{0}^{1} e^x \, dx\). Based on the above error formulae: How many equally divided subintervals are required to guarantee that the error is smaller than \(1.2 \times 10^{-5}\), respectively? [Note: In your analysis you should replace \(\xi\) with the value that maximizes error on the interval \([0, 1]\).]

A: Unanswered  C: 101; 32  E: 213; 4  G: 213; 64
B: 101; 4  D: 101; 64  F: 213; 32

Question 2 Approximate the definite integral

\( I = \int_{-1}^{1} e^{\tan(x)} \, dx \approx 2.61913 \)

with two methods:

(i) Simpson’s formula: \(I(f; a, b) \approx (b - a)(f(a) + 4f((a + b)/2) + f(b))/6\).

(ii) Expand the integrand as a Taylor expansion about \(x = 0\) upto and including quadratic terms, and integrate this approximation.

Which one of the following statements is true?

A: Unanswered  B: (i) is more accurate than (ii)  C: (ii) is more accurate than (i)  D: (i) and (ii) give the same (approximate) result  E: (i) and (ii) both give the exact answer  F: None of the above

Question 3 Find the weights \(w_0, w_1\) and \(w_1\) such that the quadrature rule

\( \int_{-1}^{1} f(x) \, dx \approx w_0 f(-1) + w_1 f(0) + w_2 f(1) \)

has the highest degree of precision \(d\). What are the values of \(w_0, w_1, w_1\) and \(d\)?

A: Unanswered  C: \(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}, 1\)  E: \(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\)  G: \(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 4\)
B: \(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}, 2\)  D: \(\frac{1}{3} \cdot \frac{4}{3} \cdot \frac{1}{3}, 4\)  F: \(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 3\)

2 Numerical ODEs

Question 4 Consider the ODE \(y'' - 3y' + 2y = 0\) with initial conditions \(y(0) = 1\) and \(y'(0) = 1\). Compute \(y'\) after one iteration of the forward Euler scheme with \(h = 0.01\). [Hint: First write this ODE as a system of first order ODEs]

A: Unanswered  C: 0.99  E: 1.01
B: 0.98  D: 1.00  F: 1.02
Question 5  Solve the differential equation
\[ \frac{d^2 y}{dt^2} + y = 0 \]
using Forward Euler. What condition must the time-step \( \Delta t \) satisfy, in order for the method to be stable?

- A: Unanswered
- B: \( \Delta t \leq 0.5 \)
- C: \( \Delta t \leq 1 \)
- D: \( \Delta t \leq 2 \)
- E: \( \Delta t \leq 0.1 \)
- F: Stable for any \( \Delta t \)
- G: Stable for no \( \Delta t \)

Question 6  For an initial value problem, \( y' = f(y), \quad y(x_0) = y_0 \), an explicit two step method is employed:
\[ y_{i+1} = a_0 y_i + a_1 y_{i-1} + h (b_0 f_i + b_1 f_{i-1}), \]
where \( f_i = f(y_i), \quad f_{i-1} = f(y_{i-1}) \). Given that \( a_0 = -4 \), what values of \( a_1, b_0, b_1 \) maximize the order of accuracy of this scheme? [Hint: Use Taylor expansions to get an expression for the local truncation error.]

- A: Unanswered
- B: \( a_1 = 3, \quad b_0 = 3, \quad b_1 = 1 \)
- C: \( a_1 = 3, \quad b_0 = 2, \quad b_1 = 2 \)
- D: \( a_1 = 4, \quad b_0 = 3, \quad b_1 = 1 \)
- E: \( a_1 = 4, \quad b_0 = 4, \quad b_1 = 2 \)
- F: \( a_1 = 5, \quad b_0 = 3, \quad b_1 = 1 \)
- G: \( a_1 = 5, \quad b_0 = 4, \quad b_1 = 2 \)
- H: \( a_1 = 6, \quad b_0 = 2, \quad b_1 = 2 \)

3  Optimization

Question 7  The figures below show the contour lines of a function of two variables \( f(x, y) \), as well as the iterative progress of three solution methods from an initial guess of \( x_0 = 4 \) and \( y_0 = 1 \):

1. Newton’s method
2. Steepest descent method
3. Nelder-Mead simplex method

Which figure corresponds to which solution method? [Note: for the Nelder-Mead simplex method the midpoint of the triangle is plotted in the figure.]

- A: Unanswered
- B: a-1, b-2, c-3
- C: a-1, b-3, c-2
- D: a-2, b-1, c-3
- E: a-2, b-3, c-1
- F: a-3, b-1, c-2
- G: a-3, b-2, c-1
- H: None of above
**Question 8**  Apply 1 iteration of the steepest descent method to the quadratic form

\[ Q(x) = \frac{1}{2} x^T \cdot A \cdot x - b^T \cdot x + c, \]

where

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}, \quad b = \begin{pmatrix} -4 \\ 4 \end{pmatrix}, \quad c = 1. \]

with the initial condition \( x_0 = (0, 0) \). What is the value of \( x_1 \)?

A: Unanswered  
B: \([0.7273, -0.7273]\)  
C: \([-0.7273, -0.7273]\)  
D: \([0.7273, 0.7273]\)  
E: \([-0.3253, -0.3253]\)  
F: \([0.3253, -0.3253]\)  
G: \([0.3253, 0.3253]\)  
H: \([-0.3253, 0.3253]\)

**Question 9**  The steepest descent method is applied to the quadratic form

\[ Q(x) = \frac{1}{2} x^T \cdot A \cdot x - b^T \cdot x + c, \]

where \( A, b \) and \( c \) are matrix, vector and scalar constants. Under what condition on the matrix \( A \) does the steepest descent method converge to the exact minimum in 1 iteration, from any initial condition \( x_0 \)? [Hint: If the initial search line \( x_0 + \alpha d_0 \) includes the exact minimum of \( Q(x) \), then the method will converge in 1 iteration.]

A: Unanswered  
B: \( A \) is a multiple of the identity matrix  
C: \( A \) is diagonal  
D: \( A \) is symmetric  
E: \( A \) is positive definite  
F: \( A \) has only positive eigenvalues  
G: \( A \) is equal to \( bb^T \)  
H: It never converges in 1 iteration

**Question 10**  Newton’s method for optimization in multiple dimensions results in a system of equations for \( x_{n+1} \):

\[ x_{n+1} = x_n - H(x_n)^{-1} f'(x_n), \]

where \( H(x_n) \) is a symmetric matrix, and the vector \( x = (x, y) \). For the function \( f(x) = x^2 y^2 \), and the initial condition \( x_0 = (1, 1) \), what is the value of \( y_1 \)?

A: Unanswered  
B: \(-\frac{1}{2}\)  
C: \(\frac{1}{7}\)  
D: \(\frac{1}{7}\)  
E: \(\frac{2}{3}\)  
F: \(-\frac{1}{6}\)  
G: \(\frac{1}{6}\)  
H: \(-\frac{1}{9}\)