Applied Numerical Analysis – Quiz #3

Modules 4 (integration), 5 and 6

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

1 Numerical integration

Question 1 Expressions for the errors of a *single* trapezoidal and Simpson's rule on an interval $[a, b]$ are:

$$E_{\text{trap}}(f) = \frac{(b-a)^3}{12} f''(\xi), \qquad E_{\text{simp}}(f) = \frac{(b-a)^5}{2880} f''''(\xi),$$

for some $\xi \in [a, b]$. Consider now a composite trapezoidal-rule and a composite Simpson's-rule, both used to approximate $\int_0^1 e^x dx$. Based on the above error formulae: How many equally divided subintervals are required to guarantee that the error is smaller than $\frac{1}{2} \times 10^{-5}$, respectively? [Note: In your analysis you should replace ξ with the value that maximizes error on the interval $[0, 1]$.]

A: Unanswered
B: 101; 4
C: 101; 32
D: 101; 64
E: 213; 4
F: 213; 32
G: 213; 64

Question 2 Approximate the definite integral

$$I = \int_{-1}^{1} e^{tan(x)} dx \approx 2.61913$$

with two methods:

(i) Simpson's formula: $I(f; a, b) \simeq (b-a)(f(a) + 4f((a+b)/2) + f(b))/6$.

(ii) Expand the integrand as a Taylor expansion about $x = 0$ upto and including quadratic terms, and integrate this approximation.

Which one of the following statements is true?

A: Unanswered
B: (i) is more accurate than (ii)
C: (ii) is more accurate than (i)
D: (i) and (ii) give the same (approximate) result
E: (i) and (ii) both give the exact answer
F: None of the above

Question 3 Find the weights w_0,w_1 and w_1 such that the quadrature rule

$$\int_{-1}^{1} f(x)dx \approx w_0 f(-1) + w_1 f(0) + w_2 f(1)$$

has the highest degree of precision d. What are the values of w_0,w_1,w_1 and d?

A: Unanswered
B: $\frac{1}{3}, \frac{4}{3}, \frac{1}{3}, 2$
C: $\frac{1}{3}, \frac{4}{3}, \frac{1}{3}, 3$
D: $\frac{1}{3}, \frac{4}{3}, \frac{1}{3}, 4$
E: $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 2$
F: $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 3$
G: $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 4$

2 Numerical ODEs

Question 4 Consider the ODE $y'' - 3y' + 2y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 1$. Compute y' after one iteration of the forward Euler scheme with $h = 0.01$. [Hint: First write this ODE as a system of first order ODEs]

A: Unanswered
B: 0.98
C: 0.99
D: 1.00
E: 1.01
F: 1.02

Question 5 Solve the differential equation

$$\frac{d^2y}{dt^2} + y = 0$$

using Forward Euler. What condition must the time-step Δt satisfy, in order for the method to be stable?

A: Unanswered
B: $\Delta t \leq 0.5$
C: $\Delta t \leq 1$
D: $\Delta t \leq 2$
E: $\Delta t \leq 0.1$
F: Stable for any Δt
G: Stable for no Δt

Question 6 For an initial value problem, $y' = f(y)$, $y(x_0) = y_0$, an explicit two step method is employed:

$$y_{i+1} = a_0 y_i + a_1 y_{i-1} + h(b_0 f_i + b_1 f_{i-1}),$$

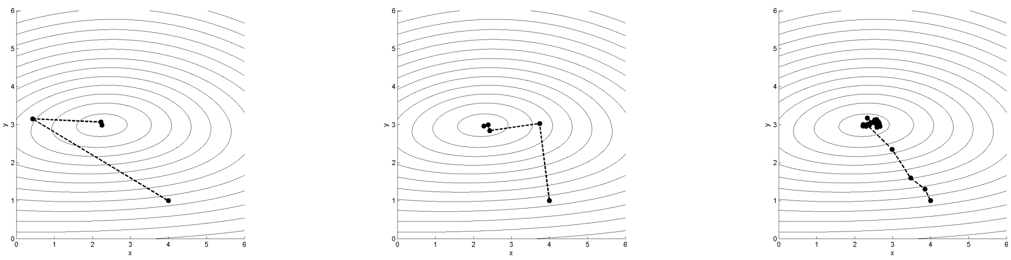
where $f_i = f(y_i)$, $f_{i-1} = f(y_{i-1})$. Given that $a_0 = -4$, what values of a_1, b_0, b_1 maximize the order of accuracy of this scheme? [Hint: Use Taylor expansions to get an expression for the local truncation error.]

A: Unanswered
B: $a_1 = 3$, $b_0 = 3$, $b_1 = 1$
C: $a_1 = 3$, $b_0 = 2$, $b_1 = 2$
D: $a_1 = 4$, $b_0 = 3$, $b_1 = 1$
E: $a_1 = 4$, $b_0 = 4$, $b_1 = 2$
F: $a_1 = 5$, $b_0 = 3$, $b_1 = 1$
G: $a_1 = 5$, $b_0 = 4$, $b_1 = 2$
H: $a_1 = 6$, $b_0 = 2$, $b_1 = 2$

3 Optimization

Question 7 The figures below show the contour lines of a function of two variables $f(x, y)$, as well as the iterative progress of three solution methods from an initial guess of $x_0 = 4$ and $y_0 = 1$:

1. Newtons method
2. Steepest descent method
3. Nelder-Mead simplex method



(a) (b) (c)

Which figure corresponds to which solution method? [Note: for the Nelder-Mead simplex method the midpoint of the triangle is plotted in the figure].

A: Unanswered
B: a-1, b-2, c-3
C: a-1, b-3, c-2
D: a-2, b-1, c-3
E: a-2, b-3, c-1
F: a-3, b-1, c-2
G: a-3, b-2, c-1
H: None of above

Question 8 Apply 1 iteration of the steepest descent method to the quadratic form

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \cdot A \cdot \mathbf{x} - \mathbf{b}^T \cdot \mathbf{x} + c,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}, \quad b = \begin{pmatrix} -4 \\ 4 \end{pmatrix}, \quad c = 1.$$

with the initial condition $x_0 = (0, 0)$. What is the value of x_1?

A: Unanswered
B: [0.7273, −0.7273]
C: [−0.7273, −0.7273]
D: [0.7273, 0.7273]
E: [−0.7273, 0.7273]
F: [0.3253, −0.3253]
G: [−0.3253, −0.3253]
H: [0.3253, 0.3253]

Question 9 The steepest descent method is applied to the quadratic form

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \cdot A \cdot \mathbf{x} - \mathbf{b}^T \cdot \mathbf{x} + c,$$

where A, $\mathbf{b}$ and c, are matrix, vector and scalar constants. Under what condition on the matrix A does the steepest descent method converge to the exact minimum in 1 iteration, from *any* initial condition $\mathbf{x}_0$? [Hint: If the initial search line $x_0 + \alpha d_0$ includes the exact minimum of $Q(\mathbf{x})$, then the method will converge in 1 iteration.]

A: Unanswered
B: A is a multiple of the identity matrix
C: A is diagonal
D: A is symmetric
E: A is positive definite
F: A has only positive eigenvalues
G: A is equal to $\mathbf{b}\mathbf{b}^T$
H: It never converges in 1 iteration

Question 10 Newton's method for optimization in multiple dimensions results in a system of equations for $\mathbf{x}_{n+1}$:

$$\mathbf{x}_{n+1} = \mathbf{x}_n - H(\mathbf{x}_n)^{-1} f'(\mathbf{x}_n),$$

where $H(\mathbf{x}_n)$ is a symmetric matrix, and the vector $\mathbf{x} = (x, y)$. For the function $f(\mathbf{x}) = x^2y^2$, and the initial condition $\mathbf{x}_0 = (1, 1)$, what is the value of y_1?

A: Unanswered
B: $-\frac{1}{2}$
C: $\frac{1}{2}$
D: $\frac{1}{3}$
E: $\frac{2}{3}$
F: $-\frac{1}{6}$
G: $\frac{1}{6}$
H: $-\frac{1}{9}$