Applied Numerical Analysis – Quiz #2

Modules 3 and 4

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

1 Advanced Interpolation

Question 1 A cubic spline $s(x)$ is used to approximate a function f on the interval $[a, b]$. The interval is divided into sub-intervals based on the grid of $N + 1$ points:

$$a = x_0 < x_1 < \cdots < x_N = b,$$

How many conditions must $s(x)$ satisfy in total, to be an interpolant with appropriate smoothness? Besides that, to obtain a unique solution, some additional constraints have to be imposed. Which of the following pairs of constraints result in a unique $s(x)$?

a.	$s(x_0) = f_0,$	$s(x_n) = f_n$
b.	$s'(x_0) = f'_0,$	$s'(x_n) = f'_n$
c.	$s''(x_0) = f''_0,$	$s''(x_n) = f''_n$
d.	$s'(x_0) = s'(x_n),$	$s''(x_0) = s''(x_n)$

A: Unanswered
B: $4N$; a, b, c
C: $4N + 2$; a, c, d
D: $4N + 2$; b, c, d
E: $4N - 2$; b, c
F: $4N - 2$; b, c, d
G: $4N - 2$; a, b, c

Question 2 A cubic spline $s(x)$ with natural boundary-conditions is differentiated once to obtain $s'(x)$. Consider the statements:

1. $s'(x)$ is piecewise quadratic
2. $s'(x)$ is continous at the nodes
3. $s'(x)$ is differentiable everywhere
4. $s'(x)$ is linear on the 1st and last interval

Which of the above are true?

A: Unanswered
B: None
C: 1, 2
D: 1, 3
E: 1, 2, 3
F: 1, 2, 4
G: 1, 3, 4
H: All

Question 3 The function $f(x) = e^x + 2x - 4$ is approximated with least-squares regression, based on 21 samples of f at $x = \{0, 1, 2, \ldots, 20\}$. The regressor is a linear combination of the following 6 basis functions:

$$\varphi_0(x) = 1, \quad \varphi_3(x) = e^x,$$
$$\varphi_1(x) = x, \quad \varphi_4(x) = e^{2x},$$
$$\varphi_2(x) = x^2, \quad \varphi_5(x) = \sin x.$$

What is the error in the regressor at $x = 1/2$?

A: Unanswered
B: 0
C: 0.0001
D: 0.0002
E: 0.0003
F: 0.0004
G: 0.0005
H: 0.0006

Question 4 Rectangular (R) or triangular (T) elements are used for patch interpolation. Consider two data sets:
Set 1

x_i	0	0.5	1	0	0.5	1	0	0.5	1
y_i	0	0	0	0.5	0.5	0.5	1	1	1
f_i	0	0.3	1.2	0.7	0.2	1.5	0.8	0.6	0.9

Set 2

x_i	0	0.2	0.3	0.3	0.5	0.6	0.8	0.9	1
y_i	0	0.1	0.7	0.9	0.5	0.4	0.7	0.9	0.4
f_i	0	0.2	1.0	0.4	0.2	0.6	1.1	0.7	1.3

Assume mixing R and T elements is not allowed. What kind of elements can be used for patch interpolation of Set 1 and Set 2, respectively?

A: Unanswered
B: only R; only T
C: only R; R or T
D: only T; only T
E: only T; R or T
F: R or T; only R
G: R or T; only T
H: R or T; R or T

Question 5 Least-squares regression can *overfit* data – to overcome this problem *regularization* is introduced. Given $N+1$ data points $(x_0, f_0), \ldots, (x_N, f_N)$, and a polynomial regressor of degree M: $\phi(x) = \sum_{j=0}^{M} a_j x^j$, the problem can be formulated as

$$\min_a \left[\sum_{i=0}^{N} (f_i - \phi(x_i))^2 + \lambda \sum_{j=0}^{M} a_j^2 \right].$$

Given the nodes $x_i = (2, 3, 4)$ with respective data values $f_i = (2, 6, 4)$, and assuming $\lambda = 2$, what are the coefficients a_0 and a_1 of the linear function that solve the regularized least-squares problem?

A: Unanswered
B: $a_0 = 2.57$, $a_1 = 0.43$
C: $a_0 = 0.41$, $a_1 = 1.11$
D: $a_0 = 1.79$, $a_3 = 0.61$

2 Numerical differentiation

Question 6 In the governing equations of viscous flows (Navier-Stokes equations), second order *mixed* partial derivatives will appear. A second-order central-difference for the mixed derivative is:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{f(x+h, y+h) - f(x+h, y-h) - f(x-h, y+h) + f(x-h, y-h)}{4h^2} + \mathcal{O}(h^\alpha)$$

which is based on step-size h in both the the x- and y-coordinates. What is the truncation error of this difference formula, i.e. the value of α?

A: Unanswered
B: h^{-2}
C: h^{-1}
D: h^0
E: h^1
F: h^2
G: h^3
H: h^4

Question 7 Consider the viscous flow of air over a flat plate. At a given station in the flow direction x, the flow velocity, u, in the direction perpendicular to the plate (the y direction) is modelled by:

$$u = 482.2(1 - e^{-y/L})$$

where $L = 0.01\,\text{m}$ is the characteristic length. The viscosity coefficient $\mu = 1.7894 \times 10^{-5}\,\text{Pa}\cdot\text{s}$. Use the expression above to find u at discrete grid points equally spaced in the y-direction, with $\Delta y = 0.001$ m.

Use these discrete values to calculate the shear stress at the wall $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$ using the second-order one sided difference rule:

$$u'(y_0) = \frac{-3u(y_0) + 4u(y_0 + \Delta y) - u(y_0 + 2\Delta y)}{2\Delta y} + O(\Delta y)^2.$$

What is the error of the calculated results compared with the exact value of τ_w (N/m^2)?

A: Unanswered
B: $0.001\,\text{Nm}^{-2}$
C: $0.002\,\text{Nm}^{-2}$
D: $0.003\,\text{Nm}^{-2}$
E: $0.004\,\text{Nm}^{-2}$
F: $0.005\,\text{Nm}^{-2}$
G: $0.006\,\text{Nm}^{-2}$
H: $0.007\,\text{Nm}^{-2}$

Question 8 Given the values of a function $f(x)$ are known at $x = x_0$, $x_0 - \frac{1}{2}h$, $x_0 + 2h$, and given the fact that:

$$f''(x_0) = a_0 f(x_0 - \frac{1}{2}h) + a_1 f(x_0) + a_2 f(x_0 + 2h) + O(h^n)$$

where $n \geq 1$, what are the values of the coefficients a_0 and a_1 and the order of convergence n?

A: Unanswered
B: $a_0 = \frac{1}{h^2}$, $a_1 = 0$, $n = 2$
C: $a_0 = \frac{1}{h^2}$, $a_1 = 0$, $n = 1$
D: $a_0 = \frac{4}{5h^2}$, $a_1 = \frac{2}{h^2}$, $n = 2$
E: $a_0 = \frac{4}{5h^2}$, $a_1 = \frac{2}{h^2}$, $n = 1$
F: $a_0 = \frac{8}{5h^2}$, $a_1 = -\frac{2}{h^2}$, $n = 2$
G: $a_0 = \frac{8}{5h^2}$, $a_1 = -\frac{2}{h^2}$, $n = 1$

Question 9 A high-order central-difference formula for the 1st-derivative of $f \in C^6([a,b])$ is:

$$f'(x_i) = \frac{f(x_i - h) - 8f(x_i - \frac{h}{2}) + 8f(x_i + \frac{h}{2}) - f(x_i + h)}{6h} + \epsilon$$

Where ϵ is the truncation error. What is the value of the term in ϵ involving $f^{(5)}(x_i)$?

A: Unanswered
B: $-\frac{h^4 f^{(5)}(x_i)}{120}$
C: $-\frac{h^4 f^{(5)}(x_i)}{240}$
D: $-\frac{h^4 f^{(5)}(x_i)}{480}$
E: $-\frac{h^4 f^{(5)}(x_i)}{960}$
F: $-\frac{h^4 f^{(5)}(x_i)}{1920}$

Question 10 Consider the central-difference formula to approximate $f'(x_0)$ at $x = x_0$ using $f(x+h)$ and $f(x-h)$. What is the largest term in the truncation error for this particular scheme?

A: Unanswered
B: $2hf'$
C: hf'
D: $\frac{1}{2}hf'$
E: $\frac{1}{3!}h^2 f''$
F: $\frac{1}{3!}h^2 f'''$
G: $\frac{1}{3!}h^3 f'''$