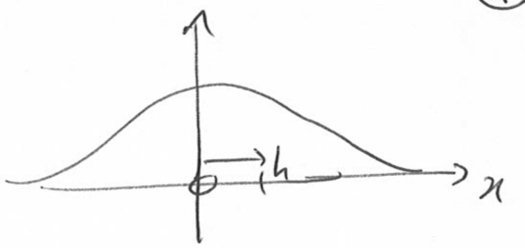
QUIZ 2018 - Part I

Q1



$$f(x) = e^{-x^2} = 1$$

$$f'(x) = -2xe^{-x^2} = 0$$

$$f''(x) = (-2+4x^2)\,e^{-x^2} = -2$$

$$f'''(x) = \left[(-2+4x^2)(-2x)+8x\right]e^{-x^2} = 0$$

$$= (-8x^3+12x)\,e^{-x^2}$$

$$f''''(x) = \left[(-24x^2+12) - 2x(-8x^3+12x)\right]e^{-x^2}$$

$$= 12$$

$x_0 = 0$

$$e^{-x^2} = 1 - \frac{2h^2}{2!} + \frac{12}{4!}h^4 + O(h^6)$$

$$= 1 - h^2 + \frac{1}{2}h^4 + O(h^6)$$

$$\int_{-1}^{1}\left(1-h^2+\frac{1}{2}h^4\right)dh = \left[h - \frac{1}{3}h^3 + \frac{1}{10}h^5\right]_{-1}^{1}$$

$$= 2 - \frac{2}{3} + \frac{1}{5} = \frac{4}{3} + \frac{1}{5}$$

F

$$= \frac{23}{15}$$

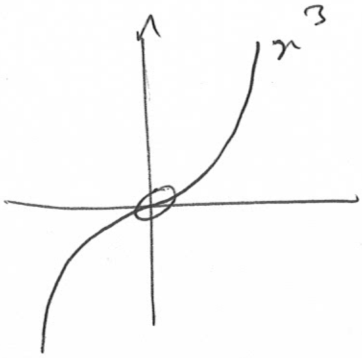
Q2

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1}{3}x_n$$

$$= \frac{2}{3}x_n$$



$$x_{n+1} = \frac{2}{3}x_n$$

$x_n \to 0$ as $n \to \infty$

so $\varepsilon_n = x_n$

$\Rightarrow$ C

Q3

$$f(x_0+h) = f(x_0) + f'(x_0)h + O(h^2)$$

$x_0 + h = x_1$

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + O(x_1 - x_0)^2$$

$$f'(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + O(x_1 - x_0)$$

$\Rightarrow$ B

OR just inspection... (FORWARD DIFFERENCES)

Q4

$$\underline{f}(\underline{x}) = \begin{pmatrix} f(x) \\ g(\underline{x}) \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 - 3 \\ 2x_1^2 + x_2^2 - 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + 2 - 3 \\ \frac{9}{2} + 1 - 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$A = \underline{f}'(\underline{x}) = \begin{bmatrix} 1 & 2 \\ 4x_1 & 2x_2 \end{bmatrix}$$

$$\underline{x}_1 = \underline{x}_0 - A(\underline{x}_0)^{-1} \underline{f}(\underline{x}_0)$$

$$= \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} - \begin{bmatrix} 1 & 2 \\ 6 & 2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \det A = -10$$

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{\det}} \quad A^{-1} = -\frac{1}{10} \begin{bmatrix} 2 & -2 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} + \frac{1}{20} \begin{bmatrix} 2 & -2 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{3}{2} \\ \frac{3}{4} \end{bmatrix}}} \Rightarrow E$$

Q5 $\quad f(x) = x^2 + \log x$

Require $|\varphi'(x)| < 1 \;\; \cancel{\forall} \;\; x \in (\tfrac{1}{2}, 1)$

1. $\varphi_1' = -\frac{1}{2} \underbrace{(-\log x)^{-\frac{1}{2}}}_{\to \infty \text{ as } x \to 1} \frac{1}{x}$ $\quad$ ✗

2. $\varphi_2' = -\underbrace{x}_{<1} 2 \underbrace{e^{-x^2}}_{<1} < 1$ ✓

3. $\varphi_3' = 1 + 2x + \frac{1}{x} > 1$ $\quad$ ✗

4. $\varphi_4' = -\frac{\frac{1}{x}x - \log x}{x^2} = \frac{\log x}{x^2} - \frac{1}{x^2}$

$x \to \frac{1}{2} \quad \frac{1}{x^2} \to 4, \quad \log x < 0 \quad \Rightarrow$ C

$\log \frac{1}{2} = -0.69\ldots$

Q6 $\quad$ Newton basis

$$\phi_0 = 1$$
$$\phi_1 = x - x_0$$
$$\phi_2 = (x - x_0)(x - x_1)$$
$$\vdots$$
$$\phi_n = (x - x_0)\ldots(x - x_{n-1})$$

Interpolation matrix

$\emptyset$

$$A = \begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \ldots & \phi_n(x_0) \\ \vdots & & & \\ \phi_0(x_n) & \ldots & \ldots & \phi_n(x_n) \end{bmatrix}$$

$\Rightarrow \det A = \phi_0(x_0) \cdot \phi_1(x_1) \cdots \phi_n(x_n)$ (product of diagonals)

$\Rightarrow$ E

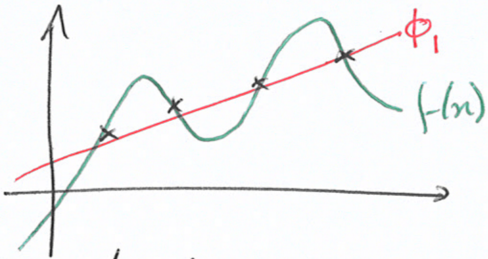
Q7

$N+1$ samples at $x_0, \dots, x_N$ of $f(x)$

$\exists \quad \phi_M(x) \in \mathbb{P}^M \qquad M = 1, 2, 3, \dots, N$

$\not\Rightarrow$ $f(x)$ is linear (affine)

$\cancel{C}$



But $\exists$ exactly 1 unique polynomial of degree N interpolating $N+1$ points

$\Rightarrow$ ϕ_M are identical (they are this unique poly)

F

Q8

$N=2 \Rightarrow p(x) = a + bx + cx^2, \quad p(2) = \cancel{a+b+}$

$= a + 2b + 4c$

$= 10 + \frac{7}{3}$

Interpolation conditions: $\quad p(x_i) = f_i$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & x_1 & x_1^2 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 14 \end{bmatrix}$$

$\Rightarrow a = 10 \quad \Rightarrow$

$3b + 9c = 4 \quad (A)$

$2b + 4c = \frac{7}{3} \quad (B)$

$x_1 b + x_1^2 c = 1 \quad (C)$

$(A) - \frac{3}{2}(B): \quad \underbrace{(3b - 3b)}_{=0} + \left(9 - \frac{3}{2} \cdot 4\right)c = 4 - \frac{7}{3} \cdot \frac{3}{2}$

$3c = \frac{1}{2}, \quad c = \frac{1}{6}$

$b = \frac{5}{6}$

in (C) $\Rightarrow$ $\underline{\underline{x_1 = 1}}$

$\Rightarrow$ D

Q9 Chebychev grid on $[-1,1]$

$$x_i = \cos\left(\frac{2i-1}{2n}\pi\right) \qquad i = 1,2,3,\dots,n$$

$n = 3$

$$x_0 = \cos\left(\pi/6\right) = \frac{\sqrt{3}}{2}$$

$$x_1 = \cos\left(\frac{\pi}{2}\right) = 0$$

$$x_2 = \cos\left(\frac{5}{6}\pi\right) = -\frac{\sqrt{3}}{2}$$

$$f_0 = \frac{\sqrt{3}^3}{4} + \frac{3}{4} + \sqrt{3} - 1$$

$$f_1 = -1$$

$$f_2 = -\frac{\sqrt{3}^3}{4} + \frac{3}{4} - \sqrt{3} - 1$$

$$\tilde{f}\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} + 1 - 1 = \frac{1}{2}$$

$x = \frac{1}{2}$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{\frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)}{\frac{\sqrt{3}}{2}\cdot\sqrt{3}}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)}{-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{+\sqrt{3}\cdot\frac{\sqrt{3}}{2}}$$

$$p\left(\frac{1}{2}\right) = l_0\left(\frac{1}{2}\right)f_0 + l_1\left(\frac{1}{2}\right)f_1 + l_2\left(\frac{1}{2}\right)f_2$$

$$= \dots = 1$$

$$\varepsilon = \left|p\left(\tfrac{1}{2}\right) - \tilde{f}\left(\tfrac{1}{2}\right)\right| = \frac{1}{2} \Rightarrow \text{D}$$

Q10

$$x = (0, 6, 8) \qquad 3 \text{ nodes} \Rightarrow p = a + bx + cx^2$$

$$y = (0, 36, 64)$$

Interpolant unique $\Rightarrow$ choice of basis irrelevant.

Notice $y_i = x_i^2 \Rightarrow f(7) = 7^2 = 49$ F