Applied Numerical Analysis – Quiz #1

Modules 1 and 2

Name: ______________________________________ Student number: ____________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

1 Root finding/Taylor approximation

Question 1 One way to approximate integrals is by first approximating the integrand by a truncated Taylor-series, and then integrating the series by hand exactly (it is just a polynomial). Using this method, approximate

$$\int_{-1}^{1} \exp(-x^2)\,\mathrm{d}x$$

by approximating the integrand with a truncated Taylor series about $x_0 = 0$, up to terms including x^4, and then integrating the series. What is the value of the approximate integral?

A: Unanswered
B: 1.1333
C: 1.2333
D: 1.3333
E: 1.4333
F: 1.5333
G: 1.6333
H: 1.7333

Question 2 Newton's method is used to find the root of $f(x) = x^3$. Given that $e_{i+1} = Ke_i^n$, where e_i is the error at iteration i, what are K and n? [Hint: Starting from the definition of the Newton iteration, derive an expression for the error.]

A: Unanswered
B: $K = \frac{1}{2}$, $N = 1$
C: $K = \frac{2}{3}$, $n = 1$
D: $K = 1$, $n = 1$
E: $K = \frac{1}{2}$, $n = 2$
F: $K = \frac{2}{3}$, $n = 2$
G: $K = 1$, $n = 2$

Question 3 One way of approximating derivatives is by rearranging a Taylor approximation. The Taylor approximation of $f(x)$ about x_0 is:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \mathcal{O}(h^3),$$

where $h := x - x_0$. Assuming the value of f is known only at x_0 and x_1, rearrange Taylor to get an approximate expression for $f'(x_0)$. What is the error in this approximation?

A: Unanswered
B: $\mathcal{O}(1)$
C: $\mathcal{O}(x_1 - x_0)$
D: $\mathcal{O}(x_1 - x_0)^2$
E: $\mathcal{O}(x_1 - x_0)^3$
F: $\mathcal{O}(x_1 - x_0)^4$
G: $\mathcal{O}(x_1 - x_0)^5$

Question 4 Consider the two-variable problem consisting of two scalar equations:

$$x_1 + 2x_2 = 3,$$
$$2x_1^2 + x_2^2 - 5 = 0.$$

Using Newton's method starting with $x_0 = (1.5, 1.0)^T$, what is the estimate of the root after single iteration? [Hint: For multiple equations and variables the derivatives form a matrix.]

A: Unanswered
B: Diverges after one step
C: $(1.5, 0.25)$
D: $(1.5, 0.5)$
E: $(1.5, 0.75)$
F: $(1.0, 0.25)$
G: $(1.0, 0.55)$
H: $(1.0, 0.75)$

Question 5 Consider the function $f(x) = x^2 + \log x$ which has a root somewhere in the interval $x \in (\frac{1}{2}, 1)$ (log is the natural logarithm). Four fixed-point iterations $x_{i+1} = \varphi(x_i)$ are applied with different choices of $\varphi(x)$:

1. $\varphi_1(x) = \sqrt{-\log x}$
2. $\varphi_2(x) = e^{-x^2}$
3. $\varphi_3(x) = x + x^2 + \log x$
4. $\varphi_4(x) = -\frac{\log x}{x}$

Given an initial guess x_0 in the (open) interval $(\frac{1}{2}, 1)$, which of these methods are *guaranteed* to converge to the root?

A: Unanswered
B: None
C: 2
D: 2,3
E: 3,4
F: 1,2,3
G: 2,3,4
H: All

2 Polynomial Interpolation

Question 6 Suppose that to interpolate a function $f(x)$, the *Newton basis* in chosen as a basis for $\mathbb{P}_n$, the vector space of polynomials of degree n. Given that the interpolation nodes are $(x_0, x_1, \ldots, x_n)$, what is the value of the determinant of the interpolation matrix $(\det(A))$?

A: Unanswered
B: $\det(A) = \sum_{i=1}^{n} \prod_{j=0}^{i} (x_i - x_j)$
C: $\det(A) = \prod_{i=1}^{n} \prod_{j=0, j\neq i}^{n} (x_i - x_j)$
D: $\det(A) = \prod_{i=1}^{n} \prod_{j=0}^{n} (x_i - x_j)$
E: $\det(A) = \prod_{i=1}^{n} \prod_{j=0}^{i-1} (x_i - x_j)$
F: $\det(A) = \sum_{i=1}^{n} \prod_{j=0}^{n} (x_i - x_j)$

Question 7 Consider an unknown function $f(x)$ which is sampled $N + 1$ times at distinct coordinates x_i. We notice that for these samples we are able to find interpolating functions $\phi_M(x)$ defined as

$$\phi_M(x) = \sum_{i=0}^{M} a_i x^i,$$

and this is possible for $M = 1$, and $M = 2$ and $\ldots$ and $M = N$. Which **one** of the following options is true?

A: Unanswered
B: $f(x)$ is constant
C: $f(x)$ is linear
D: $f(x)$ is quadratic
E: $a_0 = 0$ in all $\phi_M(x)$
F: All $\phi_M(x)$ are identical

Question 8 The nodes $x_i = (0, x_1, 3)$ and corresponding data $f_i = (10, 11, 14)$ are given. For what value of x_1 does the interpolating polynomial $p(x)$ satisfy $p(2) = 10 + \frac{7}{3}$?

A: Unanswered
B: 0.25
C: 0.5
D: 1
E: 1.5
F: 1.75
G: 2
H: 2.25

Question 9 Consider polynomial interpolation of:

$$f(x) = 2x^3 + x^2 + 2x - 1$$

on the interval $x \in [-1, 1]$, using a Chebychev grid with 3 nodes. What is the exact error $\epsilon(x) = |f(x) - p(x)|$ at $x = \frac{1}{2}$?

A: Unanswered
B: $\frac{1}{5}$
C: $\frac{1}{3}$
D: $\frac{1}{2}$
E: 1
F: $\frac{4}{3}$
G: 0

Question 10 On February 6, 2018, SpaceX successfully conducted Falcon Heavy's maiden launch. The following data of the velocity of the rocket is given as a function of time:

Time(s)	0	6	8	15	20
Velocity(m/s)	0	36	64	95	124

Interpolating the 1st 3 data points with a polynomial, employing (i) a Newton basis, and (ii) a Lagrange basis, approximate the velocity at 7 s. What approxmiately are the values?

A: Unanswered
B: (i) 37, (ii) 37
C: (i) 41, (ii) 42
D: (i) 45, (ii) 47
E: (i) 48, (ii) 48
F: (i) 49, (ii) 49
G: (i) 51, (ii) 50
H: (i) 61, (ii) 64