Applied Numerical Analysis – Resit
3 hours — Modules 1–6

Name: ____________________________ Student number: __________

DO NOT OPEN UNTIL ASKED

Instructions:
• Make sure you have a machine-readable answer form.
• Write your name and student number in the spaces above, and on the answer form.
• Fill in the answer form neatly to avoid risk of incorrect marking.
• Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
• Use only pencil on the answer form, and correct with a rubber.
• This quiz requires a calculator.
• Each question has exactly one correct answer.
• Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
• This quiz has 24 questions and 9 pages in total.
Module 1: Taylor, Root-finding, Floating-point

Question 1  One way to approximate integrals is by first approximating the integrand by a truncated Taylor-series, and then integrating the series by hand exactly. Using this method, approximate
\[ \int_0^1 \exp(x) \, dx \]
by first approximating \( \exp(x) \) with a truncated Taylor series about \( x_0 = 0 \), up to terms including \( x^3 \), and then integrating the series. What is the value of the approximate integral?

A: Unanswered  C: \( \frac{21}{24} \)  E: \( \frac{41}{24} \)
B: \( \frac{11}{24} \)  D: \( \frac{31}{24} \)  F: \( \frac{51}{24} \)

Question 2  Consider the positive floating point number system \( z = s \times 10^e \) where the base is 10. A total of 8 decimal digits are used to store \( s \) and \( e \). What is the largest number that can be represented using this number-system if a machine epsilon of at least \( 1 \times 10^{-5} \) must be achieved?

A: Unanswered  C: \( 9.99999 \times 10^5 \)  E: \( 9.99999 \times 10^7 \)
B: \( 9.9999 \times 10^4 \)  D: \( 9.99999 \times 10^6 \)  F: \( 9.99999 \times 10^8 \)
G: \( 9.99999 \times 10^9 \)  H: \( 9.99999 \times 10^9 \)

Question 3  The function \( f(x) = x^2 - 2x + 1 \) has a root at \( \tilde{x} = 1 \). A modified version of Newton’s method is used to find this root numerically. A single step of the method is:
\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \]
For which value(s) of \( C \) will this iteration have a quadratic rate of convergence for this particular \( f(x) \)? [Hint: Look for a \( C \) for which the fixed-point iteration’s \( \varphi'(x) \) is zero.]

A: Unanswered  C: \( C = \frac{1}{2} \)  E: \( C = 1 \)  G: \( C = 4 \)
B: \( C = 0 \)  D: \( C = \frac{1}{2} \)  F: \( C = 2 \)  H: \( C > 10 \)

Question 4  Consider a fixed-point iteration for a 2-dimensional root-finding problem \( f(x) = 0 \), namely
\[ x_{i+1} = \varphi(x_i), \]
where, as in the scalar case, the equation \( x = \varphi(x) \) is equivalent to \( f(x) = 0 \). Given that the error on iteration \( i \), \( \epsilon_i := x_i - \tilde{x} \) behaves like
\[ \epsilon_{i+1} = \varphi'(\xi)\epsilon_i, \]
for some (unknown) \( \xi \), under what conditions on \( \varphi' \) the Jacobian matrix of \( \varphi \) is this problem guaranteed to converge – for an initial guess close-enough to a root? [Note: \( \lambda_1, \lambda_2 \) are the eigenvalues of \( \varphi' \).]

A: Unanswered  D: \( |\lambda_1| < 1 \) and \( |\lambda_2| < 1 \)  G: \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \)
B: \( \lambda_1 \in \mathbb{R} \) and \( \lambda_2 \in \mathbb{R} \)  E: \( \lambda_1 \neq 0 \) or \( \lambda_2 \neq 0 \)  H: \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \)
C: \( |\lambda_1| < 1 \) or \( |\lambda_2| < 1 \)  F: \( \lambda_1 < 0 \) or \( \lambda_2 < 0 \)
Module 2: Polynomial Interpolation and Regression

**Question 5**  Interpolate a function $f(x)$ with polynomials. You choose to use the monomial, the Newton and the Lagrange basis on the same grid (nodes) to get the interpolants $p_1(x)$, $p_2(x)$ and $p_3(x)$ respectively. What can be said about $p_1(x)$, $p_2(x)$ and $p_3(x)$?

A: Unanswered  E: $p_2(x) \neq p_3(x)$
B: They are *always* different  F: $p_2(x) \neq p_1(x)$
C: They *can* be different  G: $p_1(x) \neq p_3(x)$
D: $p_1(x) = p_2(x) = p_3(x)$  H: None of the above

**Question 6**  We wish to perform regression with the approximant $\phi(x) = a_0 + a_1 \ln(a_2 x)$. In order to fit the pairs of points $(x_i, f_i)$, $i \in \{0, N\}$ we minimise the sum of squared-residuals,

$$ \psi = \sum_{i=0}^{N} (\phi(x_i) - f_i)^2, $$

by solving $\frac{\partial \psi}{\partial x} = 0$ for $a_0, a_1, a_2$. Consider the statements:

1. The system of equations is linear.
2. The system of equations can be solved using the recursive-bisection method.
3. The system of equations can be solved using a fixed-point iteration.
4. The system of equations can be solved using the Newton method.

Which of the above are true?

A: Unanswered  C: 2  E: 3, 4  G: 1, 2, 4
B: 1  D: 2, 3  F: 1, 2, 3

**Question 7**  During a wind tunnel experiment, the temperature is measured on the stagnation point of an airfoil. The following set of measurement data (as a function of the time) is available:

<table>
<thead>
<tr>
<th>t [s]</th>
<th>T [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13.1</td>
</tr>
<tr>
<td>4</td>
<td>19.7</td>
</tr>
<tr>
<td>9</td>
<td>21.3</td>
</tr>
<tr>
<td>12</td>
<td>22.5</td>
</tr>
</tbody>
</table>

The measurement data of the temperature (in degrees) has to be fitted with the following function:

$$ T(t) = a_1 \cdot \left( \frac{1}{6 + e^{-2t}} + \frac{1}{t + 7} \right) + a_2 \cdot \sqrt{t} $$

Note that the initial temperature is known and equal to *exactly* 7 deg. Use a combination of interpolation and regression to approximate the data. What is the value of $a_1$?

A: Unanswered  C: $a_1 = \frac{49}{2}$  E: $a_1 = \frac{69}{2}$
B: $a_1 = \frac{39}{2}$  D: $a_1 = \frac{59}{2}$  F: $a_1 = \frac{79}{2}$
Consider the function \( f(x) = \cos(x - \pi/2) \). Assume we know the value of the function only at 3 equidistant points: \((x_0, x_1, x_2) = (-h, 0, h)\). We interpolate this function using a second degree polynomial. We know an upper bound for the interpolation error for a degree \( N \) polynomial from Cauchy’s theorem:

\[
|f(x) - p_N(x)| \leq \max_{\xi \in [a,b]} |f^{N+1}(\xi)| \frac{|\omega_{N+1}(x)|}{(N+1)!},
\]

where \( \omega \) is the nodal polynomial. Based on this result, what is a bound on the error in the specific case above?

A: Unanswered
B: \( h^2 \)
C: \( h^2/(4\sqrt{2}) \)
D: \( h^3/18 \)
E: \( h^3/(9\sqrt{3}) \)
F: \( h^3/(18\sqrt{3}) \)
Module 3: Advanced interpolation

Question 9  Which of the following are shape functions (i.e. basis functions) of a linear interpolator on the triangle in 2d, with vertices \((x_1, y_1) = (1, 1), (x_2, y_2) = (1, 3),\) and \((x_3, y_3) = (3, 1)\)? [Note: Shape functions must take the value 1 at one vertex, 0 at others.]

A: Unanswered
B: \(\frac{1}{3}[1 \quad x \quad y] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix}\)
C: \(-\frac{1}{2}[1 \quad x \quad y] \cdot \begin{bmatrix} -4 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\)
D: \(-\frac{1}{2}[1 \quad x \quad xy] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix}\)
E: \(\frac{1}{2}[1 \quad xy \quad x] \cdot \begin{bmatrix} 4 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\)

Question 10  Consider the function \(f(x, y) = \frac{3}{x+y+1}\). Evaluate this function at the nodes given in the table, and linearly interpolate (triangular patch). What is the value of the interpolant at the point \((x, y) = (1/2, 1/2)\)?

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_i)</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(y_i)</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

A: Unanswered  C: \(\frac{1}{2}\)  E: \(\frac{3}{2}\)  G: \(\frac{5}{2}\)
B: 0  D: 1  F: 2  H: 3

Question 11  We interpolate \(f(x, y)\) at \(N + 1\) points \((x_i, y_i)\), all lying on a circle of radius \(R\), using radial basis-function interpolation, with the radial function \(\phi(r) = \exp(-r^4)\). The resulting interpolant takes the value \(M\) at the center of the circle. What is the sum of the unknown coefficients of the radial-basis interpolant, \(\sum_{i=0}^{N} a_i\)?

A: Unanswered  E: \(M \exp(R^4)\)
B: \(N\)  F: \(M \exp(-R^4)\)
C: \(M \exp(R^2)\)  G: \(\exp(-NR^4)\)
D: \(M \exp(-R^2)\)  H: Insufficient information
Module 4: Numerical differentiation and Integration

**Question 12**  Given the following numerical differentiation schemes:

\[
\begin{align*}
    f'(x_0) &= \frac{f(x_0 + h) - f(x_0)}{h} + O(h) \\
    f'(x_0) &= \frac{f(x_0) - f(x_0 - h)}{h} + O(h) \\
    f'(x_0) &= \frac{f(x_0 + h) - f(x_0 - h)}{2h} + O(h^2) \\
    f'(x_0) &= \frac{1}{2h} \left[ -3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) \right] + O(h^2) \\
    f'(x_0) &= \frac{1}{2h} \left[ 3f(x_0) - 4f(x_0 - h) + f(x_0 - 2h) \right] + O(h^2) \\
    f'(x_0) &= \frac{1}{12h} \left[ f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h) \right] + O(h^4).
\end{align*}
\]

Use one of these formulas to determine, as accurately as possible, an approximation for \(f'(0.0)\) using only the data in the following table:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(0.0)</th>
<th>(0.2)</th>
<th>(0.4)</th>
<th>(0.6)</th>
<th>(0.8)</th>
<th>(1.0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>0.00</td>
<td>0.01</td>
<td>0.04</td>
<td>0.09</td>
<td>0.16</td>
<td>0.25</td>
</tr>
</tbody>
</table>

What is the value of the approximation?

A: Unanswered  
B: \(-\frac{3}{10}\)  
C: \(-\frac{2}{10}\)  
D: \(-\frac{1}{10}\)  
E: 0  
F: \(\frac{1}{10}\)  
G: \(\frac{2}{10}\)  
H: \(\frac{3}{10}\)

**Question 13**  Given the numerical differentiation schemes and data in Question 12, use one of these formulas to determine, as accurately as possible, an approximation for \(f'(0.4)\). What is this approximation?

A: Unanswered  
B: \(-\frac{3}{10}\)  
C: \(-\frac{2}{10}\)  
D: \(-\frac{1}{10}\)  
E: 0  
F: \(\frac{1}{10}\)  
G: \(\frac{2}{10}\)  
H: \(\frac{3}{10}\)

**Question 14**  Consider the following integral:

\[
I = \int_0^1 \sin(\pi x) \, dx
\]

approximated by the trapezoidal rule. How many sub-intervals does the interval \([0, 1]\) need to be divided into to evaluate \(I\) with an error < \(1 \times 10^{-2}\)?

A: Unanswered  
B: 0  
C: 1  
D: 2  
E: 4  
F: 8
**Question 15**  Consider the following quadrature rule:

\[ \int_{-1}^{1} f(x) \, dx \simeq a_1 f(-1) + 4f(x_2) + a_3 f(1) \]

What is the absolute value of \( x_2 \) if the degree of precision of the quadrature rule is 2?

A: Unanswered  
B: 0  
C: \( \sqrt{1/3} \)  
D: \( \sqrt{1/2} \)  
E: \( \sqrt{2/3} \)  
F: 1

**Question 16**  Consider the following differentiation rule, where \( h \) represent the length of the discretization interval:

\[ D[f](x_i) = \frac{1}{2h} [a f(x_i) + 4f(x_{i+1}) + c f(x_{i+2})] . \]

What values of \( a \) and \( c \) are needed order to obtain a second-order accurate (i.e. truncation-error \( \sim O(h^2) \)) scheme?

A: Unanswered  
B: -3 and -1  
C: -3 and 1  
D: -1 and -3  
E: -3 and 1  
F: -3 and 3
Module 5: Numerical solution of ODEs

Note: Throughout this quiz we consider the standard form of the ODE to be:

\[ y'(t) = f(y(t)). \]

Question 17  Consider the system of two non-linear first-order ODEs

\[ z_1' = z_2, \quad z_2' = -\frac{g}{l}\sin(z_1) \]

describing an ideal pendulum, in which \( z_1 \) is the phase angle and \( z_2 \) is its time derivative. Use forward-Euler and the initial condition \( (z_1^{(0)}, z_2^{(0)}) = \left( \frac{\pi}{2}, 0 \right) \) (superscripts indicate the timestep). Assume \( \frac{g}{l} = 10 \). What is the value of \( (z_1^{(1)}, z_2^{(1)}) \) if \( \Delta t = 0.001 \)?

A: Unanswered  C: \( (\pi/2, 0.01) \)  E: \( (-\pi/2, 0.01) \)  G: \( (0, \pi/2) \)
B: \( (\pi/2, -0.01) \)  D: \( (-\pi/2, -0.01) \)  F: \( (0, -\pi/2) \)

Question 18  If the error introduced per step of an unspecified stable time integration scheme is \( O(\Delta t^4) \), what can you say about the error in the solution at a fixed time \( T \)?

A: Unanswered  C: \( O(\Delta t^1) \)  E: \( O(\Delta t^3) \)  G: \( O(\Delta t^5) \)
B: Not enough data  D: \( O(\Delta t^2) \)  F: \( O(\Delta t^4) \)  H: \( O(\Delta t^6) \)

Question 19  Consider numerical stability of the 2-step Adam-Bashford scheme, given by:

\[ y_{n+2} = y_{n+1} + \frac{\Delta t}{2} [3f_{n+1} - f_n]. \]

As usual define \( z = \lambda \Delta t \in \mathbb{C} \). Derive the discrepancy equation, and solve it using solutions of the form \( \delta_i = \beta \). What is the equation relating \( \beta \) and \( z \) that defines the stability region? [Hint: If you don’t remember how to get the discrepancy equation, use \( f_i = \lambda y_i \) in the difference scheme.]

A: Unanswered  C: \( \beta - 1 + z - 3z = 0 \)  E: \( \beta - (1 + \frac{3}{2}z)\beta + \frac{3}{2} = 0 \)
B: \( \beta^2 - (1 + \frac{3}{2}z)\beta + \frac{3}{2} = 0 \)  D: \( \beta^2 + (1 + 3z)\beta + z = 0 \)  F: \( \beta^2 - (1 - \frac{3}{2}z)\beta + \frac{3}{2} = 0 \)

Question 20  Consider the following scheme:

\[ y_{n+1} = y_n + \Delta t[\alpha f(y_n) + (1 - \alpha)f(y_{n+1})], \quad 0 \leq \alpha \leq 1 \]

which can be regarded as a generalized Heun scheme (for \( \alpha = \frac{1}{2} \) it is exactly Heun). Note that, for all \( \alpha \), the stability boundary of this scheme is a circle in the \( z \)-plane (for the standard definition of \( z \)), with center on the real-line. For what range of \( \alpha \) is the method stable for \( z = -10 \)?

A: Unanswered  D: \( \alpha < \frac{3}{4} \)  G: No such value of \( \alpha \)
B: \( \alpha < \frac{1}{2} \)  E: \( \alpha < \frac{3}{4} \)
C: \( \alpha < \frac{3}{4} \)  F: \( \alpha = 0 \) or \( \alpha = \frac{1}{2} \)
Module 6: Numerical optimization

Question 21  We want to obtain the minimum of the Himmelblau function given by:

\[ f(x, y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2. \]

A semi-analytical approach is to set the first parentheses to zero by setting \( y = 11 - x^2 \). The variable \( y \) is eliminated giving:

\[ g(x) = x^4 - 22x^2 + x + 114. \]

Apply 1 iteration of Newton’s method with \( x_0 = 2 \). What is the value of \( x_1 \)?

A: Unanswered  
B: 2 + \( \frac{33}{2} \)  
C: 2 - \( \frac{33}{2} \)  
D: 2 + \( \frac{14}{3} \)  
E: 2 - \( \frac{14}{3} \)  
F: 2 + \( \frac{55}{4} \)  
G: 2 - \( \frac{55}{4} \)

Question 22  Consider the following function:

\[ f(x, y) = x^2 + y^2 \]

that presents a minimum at \( x = (0, 0) \). Starting from any \( x_0 \neq (0, 0) \), \( M \) is the number of steepest-descent iterations and \( N \) is the number of Newton iterations needed to find the optimum exactly. Which one of the following statements is true?

A: Unanswered  
B: \( M = 1, N = 1 \)  
C: \( M > 1, N = 1 \)  
D: \( M \geq 1, N = 2 \)  
E: \( M = 1, N > 1 \)  
F: \( M \geq 1, N > 1 \)

Question 23  We want to approximate \( \pi \). We know that

\[ \pi = \arg \min_{1 \leq x \leq 5} \cos x. \]

Apply 1 iteration of golden-section search. What is the midpoint of the interval after this single iteration?

A: Unanswered  
B: 3.142  
C: 3.264  
D: 3.364  
E: 3.464  
F: 3.564  
G: 3.664  
H: 3.764

Question 24  Quadratic forms always have a single stationary point, which may be a minimum, maximum, or saddle point. Consider the quadratic form

\[ Q(x) = \frac{1}{2} x^T \cdot A \cdot x + b^T \cdot x + c. \]

Under what condition does this function have a maximum? [Hint: If you get stuck consider first 1d and then 2d examples.]

A: Unanswered  
B: \( A \) is a multiple of the identity matrix  
C: \( A \) is diagonal  
D: \( A \) is symmetric  
E: \( A \) is positive definite  
F: \( (-A) \) is positive definite  
G: \( A \) is equal to \( bb^T \)