Applied Numerical Analysis – Resit

3 hours — Modules 1–6

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **24 questions** and **9 pages** in total.

Module 1: Taylor, Root-finding, Floating-point

Question 1 One way to approximate integrals is by first approximating the integrand by a truncated Taylor-series, and then integrating the series by hand exactly. Using this method, approximate

$$\int_0^1 \exp x \,\mathrm{d}x$$

by first approximating $\exp x$ with a truncated Taylor series about $x_0 = 0$, up to terms including x^3, and then integrating the series. What is the value of the approximate integral?

A: Unanswered
B: $\frac{11}{24}$
C: $\frac{21}{24}$
D: $\frac{31}{24}$
E: $\frac{41}{24}$
F: $\frac{51}{24}$

Question 2 Consider the positive floating point number system $z = s \times 10^e$ where the base is 10. A total of 8 decimal digits are used to store s and e. What is the largest number that can be represented using this number-system if a machine epsilon of at least 1×10^{-5} must be achieved?

A: Unanswered
B: 9.99999×10^4
C: 9.99999×10^5
D: 9.99999×10^6
E: 9.99999×10^7
F: 9.99999×10^8
G: 9.99999×10^9
H: 9.99999×10^{99}

Question 3 The function $f(x) = x^2 - 2x + 1$ has a root at $\tilde{x} = 1$. A modified version of Newton's method is used to find this root numerically. A single step of the method is:

$$x_{i+1} = x_i - C\frac{f(x_i)}{f'(x_i)}.$$

For which value(s) of C will this iteration have a *quadratic* rate of convergence for this particular $f(x)$? [Hint: Look for a C for which the fixed-point iteration's $\varphi'(x)$ is zero.]

A: Unanswered
B: $C = 0$
C: $C = \frac{1}{4}$
D: $C = \frac{1}{2}$
E: $C = 1$
F: $C = 2$
G: $C = 4$
H: $C > 10$

Question 4 Consider a fixed-point iteration for a 2-dimensional root-finding problem $\boldsymbol{f}(\boldsymbol{x}) = 0$, namely

$$\boldsymbol{x}_{i+1} = \varphi(\boldsymbol{x}_i),$$

where, as in the scalar case, the equation $\boldsymbol{x} = \varphi(\boldsymbol{x})$ is equivalent to $\boldsymbol{f}(\boldsymbol{x}) = 0$. Given that the error on iteration i, $\epsilon_i := \boldsymbol{x}_i - \tilde{\boldsymbol{x}}$ behaves like

$$\epsilon_{i+1} = \varphi'(\xi)\epsilon_i,$$

for some (unknown) ξ, under what conditions on φ' the Jacobian matrix of φ is this problem guaranteed to converge – for an initial guess close-enough to a root? [Note: λ_1, λ_2 are the eigenvalues of φ'.]

A: Unanswered
B: $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}$
C: $|\lambda_1| < 1$ or $|\lambda_2| < 1$
D: $|\lambda_1| < 1$ and $|\lambda_2| < 1$
E: $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$
F: $\lambda_1 < 0$ or $\lambda_2 < 0$
G: $\lambda_1 < 0$ and $\lambda_2 < 0$
H: $\lambda_1 > 0$ and $\lambda_2 < 0$

Module 2: Polynomial Interpolation and Regression

Question 5 Interpolate a function $f(x)$ with polynomials. You choose to use the monomial, the Newton and the Lagrange basis on the same grid (nodes) to get the interpolants $p_1(x)$, $p_2(x)$ and $p_3(x)$ respectively. What can be said about $p_1(x)$, $p_2(x)$ and $p_3(x)$?

A: Unanswered
B: They are *always* different
C: They *can* be different
D: $p_1(x) = p_2(x) = p_3(x)$
E: $p_2(x) \neq p_3(x)$
F: $p_2(x) \neq p_1(x)$
G: $p_1(x) \neq p_3(x)$
H: None of the above

Question 6 We wish to perform regression with the approximant $\phi(x) = a_0 + a_1 \ln(a_2 x)$. In order to fit the pairs of points (x_i, f_i), $i \in \{0, N\}$ we minimise the sum of squared-residuals,

$$\psi = \sum_{i=0}^{N} (\phi(x_i) - f_i)^2,$$

by solving $\frac{\partial \psi}{\partial a} = 0$ for a_0,a_1,a_2. Consider the statements:

1. The system of equations is linear.
2. The system of equations can be solved using the recursive-bisection method.
3. The system of equations can be solved using a fixed-point iteration.
4. The system of equations can be solved using the Newton method.

Which of the above are true?

A: Unanswered
B: 1
C: 2
D: 2, 3
E: 3, 4
F: 1, 2, 3
G: 1, 2, 4

Question 7 During a wind tunnel experiment, the temperature is measured on the stagnation point of an airfoil. The following set of measurement data (as a function of the time) is available:

t [s]	T [deg]
1	13.1
4	19.7
9	21.3
12	22.5

The measurement data of the temperature (in degrees) has to be fitted with the following function:

$$T(t) = a_1 \cdot \left(\frac{1}{6 + e^{-2t}} + \frac{1}{t+7}\right) + a_2 \cdot \sqrt{t}$$

Note that the initial temperature is known and equal to *exactly* 7 deg. Use a combination of interpolation and regression to approximate the data. What is the value of a_1?

A: Unanswered
B: $a_1 = \frac{39}{2}$
C: $a_1 = \frac{49}{2}$
D: $a_1 = \frac{59}{2}$
E: $a_1 = \frac{69}{2}$
F: $a_1 = \frac{79}{2}$

Question 8 Consider the function $f(x) = \cos(x - \frac{\pi}{2})$. Assume we know the value of the function only at 3 equidistant points: $(x_0, x_1, x_2) = (-h, 0, h)$. We interpolate this function using a second degree polynomial. We know an upper bound for the interpolation error for a degree N polynomial from Cauchy's theorem:

$$|f(x) - p_N(x)| \leq \max_{\xi \in [a,b]} |f^{N+1}(\xi)| \frac{|\omega_{N+1}(x)|}{(N+1)!},$$

where ω is the nodal polynomial. Based on this result, what is a bound on the error in the specific case above?

A: Unanswered
B: h^2
C: $h^2/(4\sqrt{2})$
D: $h^3/18$
E: $h^3/(9\sqrt{3})$
F: $h^3/(18\sqrt{3})$

Module 3: Advanced interpolation

Question 9 Which of the following are shape functions (i.e. basis functions) of a linear interpolator on the triangle in 2d, with vertices $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (1, 3)$, and $(x_3, y_3) = (3, 1)$? [Note: Shape functions must take the value 1 at one vertex, 0 at others.]

A: Unanswered

B: $\frac{1}{3}\begin{bmatrix} 1 & x & y \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix}$

C: $-\frac{1}{2}\begin{bmatrix} 1 & x & y \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

D: $-\frac{1}{4}\begin{bmatrix} 1 & x & xy \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix}$

E: $\frac{1}{2}\begin{bmatrix} 1 & xy & x \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

Question 10 Consider the function $f(x, y) = 3/(x + y + 1)$. Evaluate this function at the nodes given in the table, and linearly interpolate (triangular patch). What is the value of the interpolant at the point $(x, y) = (1/2, 1/2)$?

i	1	2	3
x_i	0	1	1
y_i	0	0	1

A: Unanswered
B: 0
C: $\frac{1}{2}$
D: 1
E: $\frac{3}{2}$
F: 2
G: $\frac{5}{2}$
H: 3

Question 11 We interpolate $f(x, y)$ at $N + 1$ points (x_i, y_i), all lying on a circle of radius R, using radial basis-function interpolation, with the radial function $\phi(r) = \exp(-r^4)$. The resulting interpolant takes the value M at the center of the circle. What is the sum of the unknown coefficients of the radial-basis interpolant, $\sum_{i=0}^{N} a_i$?

A: Unanswered
B: N
C: $M \exp(R^2)$
D: $M \exp(-R^2)$
E: $M \exp(R^4)$
F: $M \exp(-R^4)$
G: $\exp(-NR^4)$
H: Insufficient information

Module 4: Numerical differentiation and Integration

Question 12 Given the following numerical differentiation schemes:

$$
\begin{aligned}
f'(x_0) &= \frac{f(x_0+h)-f(x_0)}{h} + O(h)\\
f'(x_0) &= \frac{f(x_0)-f(x_0-h)}{h} + O(h)\\
f'(x_0) &= \frac{f(x_0+h)-f(x_0-h)}{2h} + O(h^2)\\
f'(x_0) &= \frac{1}{2h}\Big[-3\,f(x_0)+4f(x_0+h)-f(x_0+2h)\Big] + O(h^2)\\
f'(x_0) &= \frac{1}{2h}\Big[3f(x_0)-4f(x_0-h)+f(x_0-2h)\Big] + O(h^2)\\
f'(x_0) &= \frac{1}{12h}\Big[f(x_0-2h)-8f(x_0-h)+8f(x_0+h)-f(x_0+2h)\Big] + O(h^4).
\end{aligned}
$$

Use one of these formulas to determine, as accurately as possible, an approximation for $f'(0.0)$ using only the data in the following table:

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.00	0.01	0.04	0.09	0.16	0.25

What is the value of the approximation?

A: Unanswered
B: $-\frac{3}{10}$
C: $-\frac{2}{10}$
D: $-\frac{1}{10}$
E: 0
F: $\frac{1}{10}$
G: $\frac{2}{10}$
H: $\frac{3}{10}$

Question 13 Given the numerical differentiation schemes and data in Question 12, use one of these formulas to determine, as accurately as possible, an approximation for $f'(0.4)$. What is this approximation?

A: Unanswered
B: $-\frac{3}{10}$
C: $-\frac{2}{10}$
D: $-\frac{1}{10}$
E: 0
F: $\frac{1}{10}$
G: $\frac{2}{10}$
H: $\frac{3}{10}$

Question 14 Consider the following integral:

$$I = \int_0^1 \sin(\pi x)\,\mathrm{d}x$$

approximated by the trapezoidal rule. How many sub-intervals does the interval $[0, 1]$ need to be divided into to evaluate I with an error $< 1 \times 10^{-2}$?

A: Unanswered
B: 0
C: 1
D: 2
E: 4
F: 8

Question 15 Consider the following quadrature rule:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \simeq a_1 f(-1) + 4f(x_2) + a_3 f(1)$$

What is the absolute value of x_2 if the degree of precision of the quadrature rule is 2?

A: Unanswered
B: 0
C: $\sqrt{1/3}$
D: $\sqrt{1/2}$
E: $\sqrt{2/3}$
F: 1

Question 16 Consider the following differentiation rule, where h represent the length of the discretization interval:

$$D[f](x_i) = \frac{1}{2h}\left[af(x_i) + 4f(x_{i+1}) + cf(x_{i+2})\right].$$

What values of a and c are needed order to obtain a second-order accurate (i.e. truncation-error $\sim O(h^2)$) scheme?

A: Unanswered
B: -3 and -1
C: -3 and 1
D: -1 and -3
E: -3 and 1
F: -3 and 3

Module 5: Numerical solution of ODEs

Note: Throughout this quiz we consider the standard form of the ODE to be:

$$y'(t) = f(y(t)).$$

Question 17 Consider the system of two non-linear first-order ODEs

$$z_1' = z_2 \qquad z_2' = -\frac{g}{l}\sin(z_1)$$

describing an ideal pendulum, in which z_1 is the phase angle and z_2 is its time derivative. Use forward-Euler and the initial condition $(z_1^{(0)}, z_2^{(0)}) = (\frac{\pi}{2}, 0)$ (superscripts indicate the timestep). Assume $\frac{g}{l} = 10$. What is the value of $(z_1^{(1)}, z_2^{(1)})$ if $\Delta t = 0.001$?

A: Unanswered
B: $(\pi/2, -0.01)$
C: $(\pi/2, 0.01)$
D: $(-\pi/2, -0.01)$
E: $(-\pi/2, 0.01)$
F: $(0, -\pi/2)$
G: $(0, \pi/2)$

Question 18 If the error introduced per step of an *unspecified* stable time integration scheme is $\mathcal{O}(\Delta t^4)$, what can you say about the the error in the solution at a fixed time T?

A: Unanswered
B: Not enough data
C: $\mathcal{O}(\Delta t^1)$
D: $\mathcal{O}(\Delta t^2)$
E: $\mathcal{O}(\Delta t^3)$
F: $\mathcal{O}(\Delta t^4)$
G: $\mathcal{O}(\Delta t^5)$
H: $\mathcal{O}(\Delta t^6)$

Question 19 Consider *numerical stability* of the 2-step Adam-Bashford scheme, given by:

$$y_{i+2} = y_{i+1} + \frac{\Delta t}{2}[3f_{i+1} - f_i].$$

As usual define $z = \lambda\Delta t \in \mathbb{C}$. Derive the discrepency equation, and solve it using solutions of the form $\delta_i = \beta^i$. What is the equation relating β and z that defines the stability region? [Hint: If you don't remember how to get the discrepency equation, use $f_i = \lambda y_i$ in the difference scheme.]

A: Unanswered
B: $\beta^2 - (1 + \frac{3}{2}z)\beta + \frac{z}{2} = 0$
C: $\beta - 1 + z - 3z = 0$
D: $\beta^2 + (1 + 3z)\beta + z = 0$
E: $\beta - (1 + \frac{3}{2}z)\beta + \frac{z}{2} = 0$
F: $\beta^2 - (1 - \frac{3}{2}z)\beta + \frac{1}{2} = 0$

Question 20 Consider the following scheme:

$$y_{n+1} = y_n + \Delta t[\alpha f(y_n) + (1-\alpha)f(y_{n+1})], \quad 0 \le \alpha \le 1$$

which can be regarded as a generalized Heun scheme (for $\alpha = \frac{1}{2}$ it is exactly Heun). Note that, for all α, the stability boundary of this scheme is a circle in the z-plane (for the standard definition of z), with center on the real-line. For what range of α is the method stable for $z = -10$?

A: Unanswered
B: $\alpha < \frac{1}{5}$
C: $\alpha < \frac{2}{5}$
D: $\alpha < \frac{3}{5}$
E: $\alpha < \frac{1}{2}$
F: $\alpha = 0$ or $\alpha = \frac{1}{2}$
G: No such value of α

Module 6: Numerical optimization

Question 21 We want to obtain the minimum of the Himmelblau function given by:

$$f(x, y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2.$$

A semi-analytical approach is to set the first parentesis to zero by setting $y = 11 - x^2$. The variable y is eliminated giving:

$$g(x) = x^4 - 22x^2 + x + 114.$$

Apply 1 iteration of Newton's method with $x_0 = 2$. What is the value of x_1?

A: Unanswered
B: $2 + \frac{33}{2}$
C: $2 - \frac{33}{2}$
D: $2 + \frac{44}{3}$
E: $2 - \frac{44}{3}$
F: $2 + \frac{55}{4}$
G: $2 - \frac{55}{4}$

Question 22 Consider the following function:

$$f(x, y) = x^2 + y^2$$

that presents a minimum at $\boldsymbol{x} = (0, 0)$. Starting from *any* $\boldsymbol{x}_0 \neq (0, 0)$, M is the number of steepest-descent iterations and N is the number of Newton iterations needed to find the optimum exactly. Which one of the following statements is true?

A: Unanswered
B: $M = 1$, $N = 1$
C: $M > 1$, $N = 1$
D: $M \geq 1$, $N = 2$
E: $M \geq 1$, $N > 1$
F: $M \geq 2$, $N \geq 1$

Question 23 We want to approximate π. We know that

$$\pi = \arg\min_{1 \leq x \leq 5} [\cos x].$$

Apply 1 iteration of golden-section search. What is the midpoint of the interval after this single iteration?

A: Unanswered
B: 3.142
C: 3.264
D: 3.364
E: 3.464
F: 3.564
G: 3.664
H: 3.764

Question 24 Quadratic forms always have a single stationary point, which may be a minimum, maximum, or saddle point. Consider the quadratic form

$$Q(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \cdot A \cdot \boldsymbol{x} + \boldsymbol{b}^T \cdot \boldsymbol{x} + c.$$

Under what condition does this function have a **maximum**? [Hint: If you get stuck consider first 1d and then 2d examples.]

A: Unanswered
B: A is a multiple of the identity matrix
C: A is diagonal
D: A is symmetric
E: A is positive definite
F: $(-A)$ is positive definite
G: A is equal to $\boldsymbol{b}\boldsymbol{b}^T$