Applied Numerical Analysis – Quiz #2

Modules 3 and 4

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

Question 1 Consider i) interpolation and ii) regression of a two-dimensional function $f(\mathbf{x})$ where $f : \mathbb{R}^2 \to \mathbb{R}$. Assume that the approximating function $p(\mathbf{x})$ in both cases is a linear combination of N distinct basis functions $\phi_i(\mathbf{x})$, and that f is sampled at M distinct locations. The interpolation conditions for i) lead to a system of linear equations, with system matrix A.
Consider the existence and uniqueness of solutions to these two problems. What condition is required on i) and ii) respectively, to guarantee unique solutions for each problem?

A: Unanswered
B: i) When $M = N$, and ii) when $M \geq N$.
C: i) When $\det(A) \neq 0$, and ii) $\det(A^T A) \neq 0$.
D: i) When $\det(A) \geq 0$, and ii) $\det(A^T A) \geq 0$.
E: i) Polynomial ϕ_i, and ii) polynomial ϕ_i.
F: i) Radial basis functions ϕ_i, and ii) polynomial ϕ_i.
G: i) A invertable, and ii) A symmetric.

Question 2 Steady laminar flow through a channel with parallel walls has a *parabolic* velocity profile ("Poiseuille" flow). In a channel of unit height on the interval $y \in [0, 1]$ we have two *approximate* measurements of the velocity, at $y_0 = \frac{1}{3}$ and $y_1 = \frac{2}{3}$ with values $v_0 = 7.3$ and $v_1 = 8.7$. In addition we know that the velocity at both channel walls is *exactly* zero (by the no-slip boundary condition), i.e. $v(0) = v(1) = 0$. Solve a mixed interpolation/regression problem to approximate $v(y)$, using a quadratic polynomial. What is the maximum velocity in the approximation of v?

A: Unanswered
B: 7.5
C: 8
D: 8.5
E: 9
F: 9.5
G: 10

Question 3 Consider a generalized spline interpolant $s(x)$ on the interval $[0, 10]$. The interval is divided into sub-intervals with $N + 1$ nodes:

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 10$$

at which sample data are given. Assume that the spline consists of polynomials of degree d on each interval, and that the spline is required to be $d - 1$-times continuously differentiable on the entire interval.
We impose 6 constraints at the boundaries: $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 0$, $f'(10) = 0$, $f''(10) = 0$ and $f'''(10) = 0$. What is the degree d of the polynomials needed to construct such a spline?

A: Unanswered
B: 3
C: 4
D: 5
E: 6
F: 7
G: 8
H: 9

Question 4 In polar coordinates (r, θ) (centered at the origin in the $x - y$ plane) - a function is sampled, giving

i	1	2	3	4
r_i	1	2	2	1
θ_i	0	0	$\pi/2$	$\pi/2$
f_i	2	0	0	0

at the four corners of an irregular quadralateral. Construct a patch interpolant, which is bilinear in r and θ. What is the value of the interpolant at $(x, y) = (\frac{2}{3}, \frac{2}{3})$?

A: Unanswered
B: $\pi - \frac{4}{3}\sqrt{2}$
C: $\pi - \frac{2}{3}\sqrt{2}$
D: $2\pi - \frac{4}{3}\sqrt{2}$
E: $2 - \frac{2}{3}\sqrt{2}$
F: $2 - \frac{1}{3}\sqrt{2}$
G: $3 - \frac{2}{3}\sqrt{2}$

Question 5 We interpolate $f(x, y, z)$ at $N + 1$ points (x_i, y_i, z_i), all lying on a sphere of radius R, using radial basis-function interpolation, with the radial function $\phi(r) = \exp(-r^2/\theta^2)$, where θ is a constant scaling parameter. The resulting interpolant takes the value M at the center of the sphere. What is the sum of the unknown coefficients, $\sum_{i=0}^{N} a_i$?

A: Unanswered
B: N
C: $M \exp(-R/\theta)$
D: $N \exp(-R/\theta)$
E: $M \exp(R^2/\theta^2)$
F: $N \exp(R^2/\theta^2)$
G: M
H: Not enough information.

Question 6 You are given a bivariate function $f(x, y) : [0, 1] \times [0, 1] \to \mathbb{R}$ of the form

$$f(x, y) = x^2 + y^2 + \sin(\pi x).$$

A 3×3-grid is obtained by selecting equidistant points on both axes (including the end-points). Approximate the partial derivatives of the function with respect to x and y at the central point of the domain using a forward-difference scheme. What are the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ when evaluated in this way?

A: Unanswered
B: $\frac{\partial f}{\partial x} = -0.5$ and $\frac{\partial f}{\partial y} = 1.5$
C: $\frac{\partial f}{\partial x} = 0.5$ and $\frac{\partial f}{\partial y} = -1.5$
D: $\frac{\partial f}{\partial x} = 1.5$ and $\frac{\partial f}{\partial y} = 0.5$
E: $\frac{\partial f}{\partial x} = 1.5$ and $\frac{\partial f}{\partial y} = -0.5$
F: $\frac{\partial f}{\partial x} = -1.5$ and $\frac{\partial f}{\partial y} = 1.5$

Question 7 Apply the central difference scheme:

$$\mathcal{C}(x) = \frac{f(x+h) - f(x-h)}{2h} \approx f'(x) \tag{1}$$

to evaluate the derivative of a 3rd-degree polynomial

$$f(x) = ax^3 + bx^2 + cx + d. \tag{2}$$

For any choice of step-size h, which *one* of the following is true of the error $\epsilon := f'(x) - \mathcal{C}(x)$?

A: Unanswered
B: $\epsilon > 0$
C: $\epsilon < 0$
D: $\epsilon = 0$
E: $\epsilon > 0$ for $a < 0$
F: $\epsilon > 0$ for $b < 0$
G: None of the above.

Question 8 Consider the central-difference formula:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{f^{(4)}(\xi)}{12}h^2, \quad x-h < \xi < x+h.$$

Assume that the floating-point round-off errors $\varepsilon(x-h)$, $\varepsilon(x)$, $\varepsilon(x+h)$ for $f(x-h)$, $f(x)$, and $f(x+h)$ are bounded by some number $E > 0$, and additionally that the fourth derivative of f is bounded by a number $M > 0$ in the interval $(x-h, x+h)$. Which of the following is an expression for the upper bound of the total error of the difference scheme?

A: Unanswered
B: $\frac{4E}{h^2}$
C: $\frac{4E}{h^2} + \frac{M}{12}h^2$
D: $\frac{4E}{h^2} + \frac{M}{4}h$
E: $\frac{2E}{h} + \frac{M}{2}h^2$
F: $\frac{2E}{h} + \frac{M}{4}h^2$
G: $\frac{2E}{h} + \frac{M}{12}h^2$

Question 9 We approximate the second-derivative of a function $f(x)$ with

$$f''(x) = \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} + \mathcal{O}(h)$$

Richardson extrapolation is applied recursively to increase the order of accuracy of this scheme. How many times must Richardson extrapolation be applied to reduce the truncation error to $\mathcal{O}(h^8)$?

A: Unanswered
B: 1
C: 3
D: 4
E: 5
F: 7
G: 8
H: 9

Question 10 Consider the integral:

$$I = \int_{-1}^{1} 10(e^{-x} + 1)\,\mathrm{d}x$$

What is approximately the error ε made by approximating the integral with Simpson's rule? [Hint – Simpson's rule is the quadrature rule with 3 nodes: the 2 interval end-points, and the interval center.]

A: Unanswered
B: $\varepsilon = 0.012$
C: $\varepsilon = 0.080$
D: $\varepsilon = 0.12$
E: $\varepsilon = 0.23$
F: $\varepsilon = 0.53$
G: $\varepsilon = 1.2$