Applied Numerical Analysis – Quiz #1

Modules 1 and 2

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

Question 1 For a demographic study, we need to assign one integer to each person in the world. These integers must be *consecutive*. Upper-management interference in the project means we are forced to use a floating-point variable for this person-id.
The floating point system is $z = s \times 10^e$, where $-16 \leq e \leq 15$, and $1 \leq s < 10$ with K decimal places (digits after the point). What is the minimum K needed if the world population is exactly 7.2×10^9?

A: Unanswered	C: 2	E: 7	G: 9
B: 1	D: 3	F: 8	H: 10

Question 2 The infinite sum

$$s = \sum_{n=0}^{\infty} \frac{2^n}{n!}$$

is the result of evaluating a Taylor-series about $x_0 = 0$ of one of the basic functions $\cos x$, $\sin x$, e^x or $\log(1+x)$ at a particular value of x. Given this information, what is the value of s?

A: Unanswered	C: $\cos 2$	E: e^2	G: $\log(2)$
B: 0	D: $\cos 3$	F: e^3	H: $\log(3)$

Question 3 One way to approximate integrals is by first approximating the integrand by a truncated Taylor-series, and then integrating the series by hand exactly (it is just a polynomial). Using this method, approximate

$$\int_0^{\pi/2} \sin x \, \mathrm{d}x$$

by approximating $\sin x$ with a truncated Taylor series about $x_0 = 0$, up to terms including x^3, and then integrating the series. What is the value of the approximate integral (to two decimal places)?

A: Unanswered	C: 0.52	E: 0.90	G: 1.00
B: 0	D: 0.72	F: 0.98	H: 1.02

Question 4 Consider the function

$$f(x) = \begin{cases} -x-2 & \text{if} \quad x \le -1 \\ -1 & \text{if} \ -1 < x \le \frac{1}{\pi} \\ 1 & \text{if} \quad \frac{1}{\pi} < x < 1 \\ -x+2 & \text{if} \quad x \ge 1 \end{cases}$$

A standard recursive bisection algorithm is applied to find an approximation of a root of $f(x)$ using an initial interval of $[a_0, b_0] = [-\frac{3}{2}, \frac{3}{2}]$. After a large number of iterations N, what can be said about the remaining interval $[a_N, b_N]$?

A: Unanswered.
B: Algorithm takes no steps.
C: It contains $x = -2$ (a root).
D: It contains $x = 2$ (a root).
E: It contains both $x = -2$ and $x = 2$.
F: It contains $1/\pi$ (not a root).
G: None of the above.

Question 5 Consider the function $f(x) = x + \log x$ which has a root somewhere in the interval $x \in (\frac{1}{2}, 1)$ (log is the natural logarithm). Four fixed-point iterations $x_{i+1} = \varphi(x_i)$ are applied with different choices of $\varphi(x)$:

1. $\varphi_1(x) = -\log x$
2. $\varphi_2(x) = e^{-x}$
3. $\varphi_3(x) = \frac{1}{2}(x - \log x)$
4. $\varphi_4(x) = -x - 2\log x$

Given an initial guess x_0 in the open interval $(\frac{1}{2}, 1)$, which of these methods are guaranteed to converge to the root?

A: Unanswered
B: None
C: 2
D: 2,3
E: 3,4
F: 1,2,3
G: 2,3,4
H: All

Question 6 Consider Newton's method for solving $f(x) = 0$. We have seen Newton always displays quadratic convergence if $f'(\tilde{x}) \neq 0$, with $\tilde{x}$ the exact root. Now consider a particular case where $f'(\tilde{x}) = 0$, namely $f(x) = x^2$. Assuming the initial guess is not $\tilde{x}$, how is the error from one iteration to the next related for this function?

A: Unanswered
B: $\epsilon_{i+1} = \epsilon_i^2$
C: $\epsilon_{i+1} = \frac{1}{2}\epsilon_i^2$
D: $\epsilon_{i+1} = \frac{1}{4}\epsilon_i^2$
E: $\epsilon_{i+1} = \epsilon_i$
F: $\epsilon_{i+1} = \frac{1}{2}\epsilon_i$
G: $\epsilon_{i+1} = \frac{1}{4}\epsilon_i$

Question 7 The function $f(x) = \sin(\pi x)$ is sampled at the locations $\mathbf{x} = (0, \frac{1}{2}, 1, \frac{3}{2})$. A quadratic polynomial is constructed for either the first 3 nodes or the last 3 nodes. Choose the better approximation – the choice that give the least (absolute) error at $x = \frac{1}{4}$. What is the value of this error?

A: Unanswered	C: 0.043	E: 0.21	G: 0.54
B: 0.021	D: 0.065	F: 0.46	H: 0.79

Question 8 Consider the interpolant

$$\phi(x) = \sum_{k=-K}^{K} c_k e^{ikx},$$

where $i = \sqrt{-1}$. Note that the coefficients $c_k \in \mathbb{C}$ and basis functions $\varphi_k(x) : \mathbb{R} \to \mathbb{C}$ are complex-valued, while $x \in \mathbb{R}$. You are given a set of $N + 1$ distinct nodes $0 \leq x_0 < x_1 < \ldots < x_N < 2\pi$ where $N + 1$ is odd. For what relation between K and N does there exist a unique solution to the interpolation problem for any complex-valued samples $f(x_0), \ldots f(x_N) \in \mathbb{C}$?

A: Unanswered	C: $N = 2K - 1$	E: $N = 2K + 1$	G: $2N + 1 = K$
B: $N = K$	D: $N = 2K$	F: $2N = K$	H: $2N - 1 = K$

Question 9 Consider the interpolant

$$\phi(x) = \sum_{k=0}^{n} a_k \cos(kx) \sin^2((k+1)x)$$

and the nodes $\mathbf{x} = (0, \pi/2, \pi)$. For a suitable choice of n the interpolation problem can be written $\mathbf{A}\mathbf{a} = \mathbf{f}$, where $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{R}^3$ are the interpolation coefficients and $\mathbf{f} \in \mathbb{R}^3$ are samples of a function $f(x)$ at $\mathbf{x}$. What is the value of the element $\mathbf{A}_{22}$ of the interpolation matrix?

A: Unanswered	C: $1/\sqrt{2}$	E: $\pi/2$
B: 0	D: 1	F: π

Question 10 Consider approximating $f(x) = \sin x$ on the interval $x \in [0, 2\pi]$ with a cubic polynomial (degree 3) in two different ways:

(a) A Taylor expansion about $x = 0$.

(b) An interpolant of $f(x)$ at nodes $\mathbf{x} = (0, \pi/2, 3\pi/2, 2\pi)$.

Which one of the following statements is true?

A: Unanswered
B: (a) is less accurate than (b) everywhere
C: (a) is more accurate than (b) near $x = 2\pi$
D: (a) is exact for $x < \pi/8$
E: (a) is more accurate near $x = 0$, (b) is more accurate for large x.
F: (a) is not an approximation of $f(x)$.
G: (a) does not intersect $f(x)$ at any point.