Applied Numerical Analysis – Resit

3 hours — Modules 1–6

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **24 questions** and **8 pages** in total.

Module 1: Taylor, Root-finding, Floating-point

Throughout we use $T_k[f, x_0](x)$ to denote the truncated Taylor series:

$$T_k[f, x_0](x) := \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i.$$

Question 1 Approximate $f(x) = e^x$ with two different Taylor expansions: $g(x) = T_2[f, 0](x)$ and $h(x) = T_3[f, 0](x)$, both constructed about $x = 0$. Define errors

$$\epsilon_1 = |f(\frac{1}{100}) - g(\frac{1}{100})|, \qquad \epsilon_2 = |f(1) - h(1)|.$$

Which *one* of the following is true?

A: Unanswered
B: $\varepsilon_2 = 0.0516$, $\varepsilon_2 \geq \varepsilon_1$
C: $\varepsilon_2 = 0.0516$, $\varepsilon_2 \leq \varepsilon_1$
D: $\varepsilon_2 = 0.0152$, $\varepsilon_2 \geq \varepsilon_1$
E: $\varepsilon_2 = 0.0152$, $\varepsilon_2 \leq \varepsilon_1$
F: $\varepsilon_2 = 0.1111$, $\varepsilon_2 \geq \varepsilon_1$
G: $\varepsilon_2 = 0.1111$, $\varepsilon_2 \leq \varepsilon_1$

Question 2 One way to approximate integrals is by first approximating the integrand by a truncated Taylor-series, and then integrating the series by hand exactly. Using this method, approximate

$$\int_{-\pi/2}^{\pi/2} \cos x \, \mathrm{d}x$$

by first approximating $\cos x$ with a truncated Taylor series about $x_0 = 0$, up to terms including x^2, and then integrating the series. What is the value of the approximate integral?

A: Unanswered
B: $\frac{2\pi}{3}$
C: $\frac{4\pi}{5}$
D: $\pi - \frac{\pi^3}{24}$
E: $\pi - \frac{\pi^2}{8}$
F: π
G: 2

Question 3 The small-angle approximation for trigonometric functions is based on a Taylor expansion about $x = 0$, up to quadratic terms. For which angle does approximation of $\sin(x)$ have a relative error exceeding approximately 1.0%? [Hints: Relative error is defined as $\frac{f_{true} - f_{approx}}{f_{true}} \times$ 100%. Rather than solving the resulting equation, each of the given options can be checked.]

A: Unanswered
B: $7\pi/180$
C: $14\pi/180$
D: $21\pi/180$
E: $\pi/2$
F: π

Question 4 A fixed point iteration is specified as:

$$x_{i+1} = \tan x_i - 2x_i + 1.$$

What equation is being solved here?

A: Unanswered
B: $0 = \tan x - x + 1$
C: $0 = \tan x - 2x + 1$
D: $0 = \tan x + x + 1$
E: $0 = \tan x + 1$
F: $0 = \tan x - x^2 + x$
G: $0 = \tan x - 3x + 1$

Module 2: Polynomial Interpolation and Regression

Question 5 Interpolate a function $f(x)$ with polynomials. You choose to use the monomial, the Newton and the Lagrange basis on the same grid (nodes) to get the interpolants $p_1(x)$, $p_2(x)$ and $p_3(x)$ respectively. What can be said about $p_1(x)$, $p_2(x)$ and $p_3(x)$?

A: Unanswered
B: They are *always* different
C: They *can* be different
D: $p_1(x) = p_2(x) = p_3(x)$
E: $p_2(x) \neq p_3(x)$
F: $p_2(x) \neq p_1(x)$
G: $p_1(x) \neq p_3(x)$
H: None of the above

Question 6 Consider an unknown function $f(x)$ which is sampled $N + 1$ times at distinct coordinates x_i. *For these samples* we are able to find interpolating functions $\phi_M(x)$ defined as

$$\phi_M(x) = \sum_{i=0}^{M} a_i x^i,$$

and this is possible for $M = 1$, and $M = 2$ and ... and $M = N$. Which **one** of the following options is true?

A: Unanswered
B: $f(x)$ is linear
C: $f(x)$ is quadratic
D: $a_0 = 0$ in all $\phi_M(x)$
E: All $\phi_M(x)$ are identical

Question 7 Consider the following data points:

i	0	1	2	3
t	0	1	2	3
y	0	1	4	9

We interpolate this data using the Lagrange basis:

$$y(t) \simeq \sum_{i=0}^{3} y_i \, \ell_i(t).$$

What is the Lagrange polynomial $\ell_1(t)$ in the above? (Note that the index starts at 0!)

A: Unanswered
B: $\ell_1(t) = t(t-2)(t-3)$
C: $\ell_1(t) = \frac{t(t-2)(t-3)}{2}$
D: $\ell_1(t) = \frac{t(t-2)(t-3)}{6}$
E: $\ell_1(t) = (t-1)(t-2)(t-3)$
F: $\ell_1(t) = \frac{(t-1)(t-2)(t-3)}{2}$
G: $\ell_1(t) = \frac{(t-1)(t-2)(t-3)}{6}$

Question 8 Consider the function $f(x) = \sin(x)$. Assume we know the value of the function only at 3 equidistant points: $(x_0, x_1, x_2) = (-h, 0, h)$. We interpolate this function using a second degree polynomial. We know an upper bound for the interpolation error for a degree N polynomial from Cauchy's theorem:

$$|f(x) - p_N(x)| \leq \max_{\xi \in [a,b]} |f^{N+1}(\xi)| \frac{|\omega_{N+1}(x)|}{(N+1)!},$$

where ω is the nodal polynomial. Based on this result, what is a bound on the error in the specific case above?

A: Unanswered
B: h^2
C: $h^2/(4\sqrt{2})$
D: $h^3/18$
E: $h^3/(18\sqrt{3})$
F: $h^3/(9\sqrt{3})$

Module 3: Advanced interpolation

Question 9 Which of the following functions is not a linear, quadratic or cubic spline?

A: Unanswered
B:

$$f(x) = \begin{cases} 2x^3 - \frac{9}{2}x^2 + 5x - \frac{3}{2}, & x \in [0,1] \\ 3x^3 - \frac{15}{2}x^2 + 8x - \frac{5}{2}, & x \in [1,2] \end{cases}$$

C:

$$f(x) = \begin{cases} 3x^3 - 8x^2 + 8x - 1, & x \in [0,1] \\ 2x^3 - 5x^2 + 3x + 2, & x \in [1,2] \end{cases}$$

D:

$$f(x) = \begin{cases} 2x - 1, & x \in [0,1] \\ 3x - 2, & x \in [1,2] \end{cases}$$

E:

$$f(x) = \begin{cases} 2x^2 - 2x + 1, & x \in [0,1] \\ 3x^2 - 4x + 2, & x \in [1,2] \end{cases}$$

Question 10 Consider the function $f(x, y) = 1/(x + y + 1)$. Evaluate this function at the nodes given in the table, and linearly interpolate (triangular patch). What is the value of the interpolant at the point $(x, y) = (1/2, 1/2)$?

i	1	2	3
x_i	0	1	1
y_i	0	0	1

A: Unanswered
B: $\frac{1}{2}$
C: $\frac{2}{3}$
D: $\frac{3}{4}$
E: $\frac{5}{6}$
F: $\frac{7}{8}$
G: $\frac{8}{9}$
H: 1

Question 11 Suppose we define an approximating function $\phi(x) = \sum_{i=1}^{I} a_i\, b_i(x)$, where $b_i(x)$ are given functions of x. To determine the coefficients $\{a_i : i = 1 \dots I\}$ from given data $\{f_i : i = 1 \dots N\}$ at N distinct points, we can either use interpolation or least-squares approximation. Which one of the following is true? Interpolation and least-squares approximation:

A: Unanswered
B: Always provide the same coefficients;
C: Provide the same coefficients if $N = I$ and, at the same time, the data are not noisy;
D: Provide the same coefficients if $N = I$;
E: Provide the same coefficients if the interpolation problem is uniquely solvable;
F: Provide the same coefficients if the data are not noisy;
G: Never provide the same coefficients;

Question 12 We interpolate $f(x, y)$ at $N + 1$ points (x_i, y_i), all lying on a circle of radius R, using radial basis-function interpolation, with the radial function $\phi(r) = \exp(-r^2)$. The resulting interpolant takes the value M at the center of the circle. What is the sum of the unknown coefficients of the radial-basis interpolant, $\sum_{i=0}^{N} a_i$?

A: Unanswered
B: N
C: $N \exp(R^2)$
D: $N \exp(-R^2)$
E: $M \exp(R^2)$
F: $M \exp(-R^2)$
G: $\exp(-NR^2)$
H: Insufficient information

Module 4: Numerical differentiation and Integration

Question 13 Given the following numerical differentiation schemes:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} + O(h)$$
$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} + O(h)$$
$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + O(h^2)$$
$$f'(x_0) = \frac{1}{2h}\Big[-3\,f(x_0) + 4f(x_0+h) - f(x_0+2h)\Big] + O(h^2)$$
$$f'(x_0) = \frac{1}{2h}\Big[3f(x_0) - 4f(x_0-h) + f(x_0-2h)\Big] + O(h^2)$$
$$f'(x_0) = \frac{1}{12h}\Big[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\Big] + O(h^4).$$

Use one of these formulas to determine, as accurately as possible, an approximation for $f'(0.0)$ using only the data in the following table:

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.00	0.01	0.04	0.09	0.16	0.25

What is the value of the approximation?

A: Unanswered
B: $-\frac{3}{10}$
C: $-\frac{2}{10}$
D: $-\frac{1}{10}$
E: 0
F: $\frac{1}{10}$
G: $\frac{2}{10}$
H: $\frac{3}{10}$

Question 14 Given the numerical differentiation schemes and data in Question 13, use one of these formulas to determine, as accurately as possible, an approximation for $f'(0.6)$. It is?

A: Unanswered
B: $-\frac{3}{10}$
C: $-\frac{2}{10}$
D: $-\frac{1}{10}$
E: 0
F: $\frac{1}{10}$
G: $\frac{2}{10}$
H: $\frac{3}{10}$

Question 15 Consider grid with uniform spacing h, nodes x_i, and values f_i. Higher-degree derivatives can be approximated by recursive application of difference-rules for 1st-derivatives. By repeated application of central differences $f'(x_i) \simeq \frac{f_{i+1}-f_{i-1}}{2h}$, obtain an approximation to $f''(x_i)$. What is the stencil of this approximation (i.e. the set of nodes needed to evaluate the rule)?

A: Unanswered
B: $(i-1, i+1)$
C: $(i-2, i+2)$
D: $(i-1, i, i+1)$
E: $(i-2, i, i+2)$
F: $(i-2, i-1, i+1, i+2)$
G: $(i-2, i-1, i, i+1, i+2)$

Question 16 **Hyperbolic quadrature**. Until now we have considered quadrature rules that integrate polynomials of degree d exactly. Using the same principles derive a quadrature rule that integrates the functions $\phi = [1, \sinh(x), \cosh(x)]$ exactly on the interval $x \in [-1, 1]$. For nodes use $x = (-1, 0, 1)$. What is the weight of the node at $x = 1$? [Note: $\cosh(x) = (e^x + e^{-x})/2$, $\sinh(x) = (e^x - e^{-x})/2$.]

A: Unanswered
B: $\frac{\sinh(1)}{\cosh(1)}$
C: $\frac{\cosh(1)}{\sinh(1)}$
D: $\frac{\sinh(1)+1}{\cosh(1)+1}$
E: $\frac{\cosh(1)+1}{\sinh(1)+1}$
F: $\frac{\sinh(1)-1}{\cosh(1)-1}$
G: $\frac{\cosh(1)-1}{\sinh(1)-1}$
H: No such rule exists.

Module 5: Numerical solution of ODEs

Throughout we assume the standard form of the ODE: $\mathbf{y}'(t) = f(\mathbf{y}(t))$.

Question 17 Consider the ODE $y' = -y$ with initial-condition $y(0) = 2$. Use forward-Euler to approximate $y(1)$ with a step-size of $\Delta t = 0.1$. What is approximately the error with respect to the exact solution at $t = 1$?

A: Unanswered
B: 0.016
C: 0.025
D: 0.038
E: 0.049
F: 0.081
G: 0.010

Question 18 Which of the following numerical schemes are *explicit*?

(i) $y_{n+1} = y_n + \Delta t\, f(y_n)$

(ii) $y_{n+1} = y_n + \Delta t\, f(y_{n+1})$

(iii) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(\hat{y})\right), \quad \hat{y} = y_n + \Delta t f(y_n)$

(iv) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(y_{n+1})\right)$

(v) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(y_{n-1})\right)$

A: Unanswered
B: (i), (iii)
C: (ii), (iv)
D: (iv), (v)
E: (i), (iii), (v)
F: (i), (iv), (v)
G: (i), (iii), (iv), (v)
H: (iii), (iv), (v)

Question 19 For the initial value problem $y' = cy, y(t_0) = y_0$, and $c \in \mathbb{C}$ a complex number, we consider the method:

$$y_{n+1} = y_n + \Delta t[\alpha f_n + (1-\alpha) f_{n-1}],$$

where $f_n = cy_n$. Let $z = c\Delta t$. The corresponding defect equation has solutions of the form $\delta_n = \beta^n$, and is therefore stable for $|\beta| \leq 1$. Which of the following equations relates β to z?

A: Unanswered
B: $0 = \beta^2 - (\alpha - 1)\beta - z$
C: $0 = \beta - (z^2 + z\alpha + \alpha)$
D: $0 = \beta^2(1 - \alpha^2) + 2\alpha z - 1$
E: $0 = \beta^2 - \alpha^2 z^2$
F: $0 = \beta^2(1 + \alpha z) - \alpha - z$
G: $0 = \beta - \frac{1 + z\alpha/2}{1 - z\alpha/2}$
H: $0 = \beta^2 - \beta(1 + \alpha z) - z(1 - \alpha)$

Question 20 Consider a difference scheme $\mathbf{y}_{i+1} = \mathcal{D}(\mathbf{y}_i)$ applied to an initial-value problem with exact solution $y(t)$. Let the scheme predict a discrete solution $\mathbf{y}_i$ at time t_i. Which of the following statements are true?

(i) The *global error* at time t_i is $|y(t_i) - \mathbf{y}_i|$.

(ii) The *local error* at time t_i is $|\mathcal{D}(y(t_i)) - y(t_{i+1})|$.

(iii) If $y(t)$ is a polynomial of degree ≤ 1, and $\mathcal{D}$ is 1st-order accurate, then $y(t_i) = \mathbf{y}_i$.

A: Unanswered
B: None
C: (i)
D: (ii)
E: (i) and (ii)
F: (i) and (iii)
G: (ii) and (iii)
H: (i), (ii) and (iii)

Module 6: Numerical optimization

Question 21 You are jogging on a path which can be described by a curve $y = \log x$ (where the natural logarithm is used). Your car is parked at the origin $(0, 0)$. You want to stop jogging at the point closest to your car. Use Newton's method to minimize the *squared* distance to the car, with an initial guess of $x_0 = 1$. What is the value of x_1?

A: Unanswered
B: $\frac{1}{8}$
C: $\frac{1}{6}$
D: $\frac{1}{5}$
E: $\frac{1}{4}$
F: $\frac{1}{3}$
G: $\frac{1}{2}$
H: 1

Question 22 Consider minimization of the function of two variables $\mathbf{x} = (x, y)$:

$$f(\mathbf{x}) = \frac{1}{2}(ax^2 + by^2)$$

using Newton's method. What is the value of the vector $\mathbf{x}_1 - \mathbf{x}_0$ for $\mathbf{x}_0 = (x_0, y_0) = (1, 2)$?

A: Unanswered
B: $(2, -1)$
C: $(-1, -2)$
D: $(-a, -2b)$
E: $(a, 2b)$
F: $(-2, 1)$
G: $(1, 2)$
H: $(a, -b)$

Question 23 Three iterations of the Golden-Section Search algorithm are applied to minimize the function $f(x) = (\cosh x)^2 + x - 1$ using the starting interval $[-1, 10]$. What is approximately the width of the interval after three iterations? [Hint: The golden ratio is $\phi = \frac{1}{2}(1 + \sqrt{5})$.]

A: Unanswered
B: 1.6
C: 2.6
D: 3.6
E: 4.6
F: 5.6
G: 11.0

Question 24 Quadratic forms always have a single stationary point, which may be a minimum, maximum, or saddle point. Consider the quadratic form

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \cdot A \cdot \mathbf{x} + \mathbf{b}^T \cdot \mathbf{x} + c.$$

Under what condition on A is this function guaranteed to have a *maximum*? [Hint: If you get stuck consider first 1d and then 2d examples.]

A: Unanswered
B: A is a multiple of the identity matrix
C: A is diagonal
D: A is symmetric
E: A is positive definite (all positive eigenvalues)
F: A is negative definite (all negative eigenvalues)
G: A has only positive eigenvalues
H: A is equal to $\mathbf{b}\mathbf{b}^T$