Applied Numerical Analysis – Quiz #3

Modules 5 and 6

Name: ______________________________ Student number: __________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

Module 5: ODEs

Note: Throughout this quiz we consider the standard form of the ODE to be:

$$y'(t) = f(y(t)).$$

Question 1 If the error introduced per step of an *unspecified* stable time integration scheme is $\mathcal{O}(\Delta t^3)$, what can you say about the the error in the solution at a fixed time T?

A: Unanswered
B: Not enough data
C: $\mathcal{O}(\Delta t^1)$
D: $\mathcal{O}(\Delta t^2)$
E: $\mathcal{O}(\Delta t^3)$
F: $\mathcal{O}(\Delta t^4)$
G: $\mathcal{O}(\Delta t^5)$
H: $\mathcal{O}(\Delta t^6)$

Question 2 Consider the system of two non-linear first-order ODEs

$$z_1' = z_2$$
$$z_2' = -\frac{g}{l}\sin(z_1)$$

describing an ideal pendulum, in which z_1 is the phase angle and z_2 is its time derivative. Use forward-Euler and the initial condition $(z_1^{(0)}, z_2^{(0)}) = (\frac{\pi}{2}, 0)$ (superscripts indicate the timestep). Assume $\frac{g}{l} = 10$. What is the value of $(z_1^{(1)}, z_2^{(1)})$ if $\Delta t = 0.001$?

A: Unanswered
B: $(\pi/2, -0.01)$
C: $(\pi/2, 0.01)$
D: $(-\pi/2, -0.01)$
E: $(-\pi/2, 0.01)$
F: $(0, -\pi/2)$
G: $(0, \pi/2)$

Question 3 Consider the *implicit* scheme:

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(y_{n+1}) + f(y_n)\right].$$

By linearization one can obtain an *explicit* scheme which is an approximation to this - with approximation error $O(\Delta t^3)$. Which of the following is that explicit scheme? [Note: $f_y := \frac{\partial f}{\partial y}$]

A: Unanswered
B: $y_{n+1} = y_n + \Delta t f(y_n)$
C: $y_{n+1} = y_n + \Delta t f(y_{n+1})$
D: $y_{n+1} = y_n + \Delta t[1 - 2\Delta t f_y(y_n)]^{-1} f(y_n)$
E: $y_{n+1} = y_n + \Delta t[1 - \frac{1}{2}\Delta t f_y(y_n)]^{-1} f(y_n)$
F: $y_{n+1} = y_n + 2[1 + \Delta t f_y(y_n)]^{-1} f(y_n)$
G: $y_{n+1} = y_n + 2\Delta t[1 + \Delta t f_y(y_n)]^{-1} f(y_n)$
H: $y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(y_n) + f(y_{n-1})\right]$

Question 4 Consider the ODE $y' = -2y$. Using the forward-Euler scheme we can write:

$$y_{i+1} = y_i + h(-2y_i) = y_0\,(1 - 2h)^{i+1}$$

For which range of stepsize h is the scheme stable for this equation?

A: Unanswered
B: Never stable
C: $h > 0$
D: $h < 0$
E: $h > 0.5$
F: $0 < h < 0.5$
G: $0 < h < 1$
H: $0 < h < 2$

Question 5 Consider *numerical stability* of the 2-step Adam-Bashford scheme, given by:

$$y_{i+2} = y_{i+1} + \frac{\Delta t}{2}[3f_{i+1} - f_i].$$

As usual define $z = \lambda \Delta t \in \mathbb{C}$. Derive the discrepency equation, and solve it using solutions of the form $\delta_i = \beta^i$. What is the equation relating β and z that defines the stability region? [Hint: If you don't remember how to get the discrepency equation, use $f_i = \lambda y_i$ in the difference scheme.]

A: Unanswered
B: $\beta^2 - (1 + \frac{3}{2}z)\beta + \frac{z}{2} = 0$
C: $\beta - 1 + z - 3z = 0$
D: $\beta^2 + (1 + 3z)\beta + z = 0$
E: $\beta - (1 + \frac{3}{2}z)\beta + \frac{z}{2} = 0$
F: $\beta^2 - (1 - \frac{3}{2}z)\beta + \frac{1}{2} = 0$

Question 6 Consider a difference scheme $\mathbf{y}_{i+1} = \mathcal{D}(\mathbf{y}_i)$ applied to an initial-value problem with exact solution $y(t)$. Let the scheme predict a discrete solution $\mathbf{y}_i$ at time t_i. Which of the following statements are true?

(i) The *global error* at time t_i is $|y(t_i) - \mathbf{y}_i|$.

(ii) The *local error* at time t_i is $|\mathcal{D}(y(t_i)) - y(t_{i+1})|$.

(iii) If $y(t)$ is a polynomial of degree ≤ 1, and $\mathcal{D}$ is 1st-order accurate, then $y(t_i) = \mathbf{y}_i$.

A: Unanswered
B: None
C: (i)
D: (ii)
E: (i) and (ii)
F: (i) and (iii)
G: (ii) and (iii)
H: (i), (ii) and (iii)

Module 6: Optimization

Question 7 Coca-Cola cans are cylinders of constant volume $V = 311\,\text{cm}^3$ optimized for minimum surface to volume (S/V) ratio. What is approximately the radius of a Coca-Cola can?

A: Unanswered
B: Insufficient Data For A Meaningful Answer
C: 1.25 cm
D: 3.67 cm.
E: 4.51 cm
F: 6.54 cm
G: 7.34 cm
H: 8.03 cm

Question 8 We want to obtain the minimum of the Himmelblau function given by:

$$f(x, y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2.$$

A semi-analytical approach is to set the first parentesis to zero by setting $y = 11 - x^2$. The variable y is eliminated giving:

$$g(x) = x^4 - 22x^2 + x + 114.$$

Apply 1 iteration of Newton's method with $x_0 = 2$. What is the value of x_1?

A: Unanswered
B: $2 + \frac{33}{2}$
C: $2 - \frac{33}{2}$
D: $2 + \frac{44}{3}$
E: $2 - \frac{44}{3}$
F: $2 + \frac{55}{4}$
G: $2 - \frac{55}{4}$

Question 9 A minimization problem in $f(x)$ can be reduced to a zero-finding problem by setting $f'(x) = 0$. In a similar way a 1d zero-finding problem $g(x) = 0$ can be rephrased as an minimization problem: Find x such that

$$\min_x \frac{1}{2} [g(x)]^2 .$$

This will have a global minimum at every x which satisfies $g(x) = 0$, but may also have other local minima $\tilde{x}_1, \ldots, \tilde{x}_n$. What equation(s) will these local minima satify?

A: Unanswered
B: $g(\tilde{x}_i) = 0$
C: $\sum_i g(\tilde{x}_i) = 0$
D: $\sum_i \tilde{x}_i = 0$
E: $g'(\tilde{x}_i) = 0$ and $g''(\tilde{x}_i) > 0$
F: $g'(\tilde{x}_i) = 0$ and $g''(\tilde{x}_i) < 0$
G: $g'(\tilde{x}_i) = 0$ and $g(\tilde{x}_i) \cdot g''(\tilde{x}_i) > 0$
H: $g'(\tilde{x}_i) = 0$ and $g(\tilde{x}_i) \cdot g''(\tilde{x}_i) < 0$

Question 10 Pseudo-code for the Golden-section search is given below, containing *one* error. What line of the code contains the error? [Note: $\varphi = \frac{1+\sqrt{5}}{2}$, and input is the starting interval $[a, b]$, and an error tolerance ε.]

1. $x_1 \leftarrow a$
2. $x_2 \leftarrow b$
3. $x_3 \leftarrow x_2 - \frac{1}{\varphi}(x_2 - x_1)$
4. $x_4 \leftarrow x_1 + \frac{1}{\varphi}(x_2 - x_1)$
5. error $\leftarrow \infty$
6. **while** error $> \varepsilon$
7.   **if** $f(x_3) > f(x_4)$
8. $x_2 \leftarrow x_4$
9. $x_4 \leftarrow x_3$
10. $x_3 \leftarrow x_2 - \frac{1}{\varphi}(x_2 - x_1)$
11.   **else**
12. $x_1 \leftarrow x_3$
13. $x_3 \leftarrow x_4$
14. $x_4 \leftarrow x_1 + \frac{1}{\varphi}(x_2 - x_1)$
15.   **end if**
16.   error $= |x_3 - x_4|$
17. **end while**
18. **return** x_3, x_4

A: Unanswered
B: Line 7
C: Line 10
D: Line 14
E: Line 16