Applied Numerical Analysis – Quiz #2

Modules 3 and 4

Name: ______________________________ Student number: ____________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

Module 3: Advanced Interpolation

Question 1 Do there exist constants a,b,c and d so that the function $s(x)$

$$s(x) = \begin{cases} ax^3 + x^2 + cx & -1 \le x \le 0 \\ bx^3 - x^2 + dx & 0 \le x \le 2 \end{cases}$$

is a natural cubic spline? If yes, what are the values of a, b, c and d? [Note: "Natural" implies $s'' = 0$ at the boundaries.]

A: Unanswered
B: Yes, $a = b = \frac{1}{3}, c = d$.
C: Yes, $a = \frac{1}{3}, b = \frac{1}{6}, c = d = 0$.
D: Yes, $a = \frac{1}{6}, b = \frac{1}{3}, c = d$.
E: Yes, $a = b = \frac{1}{6}, c = d = 0$.
F: No, $s(x)$ is not a spline.
G: No, $s(x)$ is a spline, but not a natural spline.

Question 2 Neglecting friction the height in time $y(t)$ of a launched projectile can be modelled as

$$y(t) = v_0 t \sin(\theta) - \frac{g}{2} t^2$$

with launching angle θ, launching velocity v_0 and gravitational acceleration $g = 10\ m/s^2$. You interpolate measurements of $y(t_i)$ sampled at regular intervals in time on the interval $[0, T]$, i.e. $t = [0, \Delta t, 2\Delta t, \dots, T]$ with a natural cubic spline. *Based on the spline* what is the value of the acceleration of the projectile at time T?

A: Unanswered
B: $g = 0$ m/s^2
C: $g = 1$ m/s^2
D: $g = 2.5$ m/s^2
E: $g = 5$ m/s^2
F: $g = 10$ m/s^2
G: $g = 25$ m/s^2
H: Insufficient information given.

Question 3 Which of the following are shape functions (i.e. basis functions) of a linear interpolator on the triangle in 2d, with vertices $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (1, 3)$, and $(x_3, y_3) = (3, 1)$? [Note: Shape functions must take the value 1 at one vertex, 0 at others.]

A: Unanswered
B: $\frac{1}{3} \begin{bmatrix} 1 & x & y \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix}$
C: $-\frac{1}{2} \begin{bmatrix} 1 & x & y \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$
D: $-\frac{1}{4} \begin{bmatrix} 1 & x & xy \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix}$
E: $\frac{1}{2} \begin{bmatrix} 1 & xy & x \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

Question 4 Consider the following statements relating to the approximating function (the “approximant”) obtained with polynomial least-squares regression:

(i) The approximant is always smooth.

(ii) In the univariate (1-dimensional) case, the approximant is unique.

(iii) The approximant minimizes the sum of the squares of the residuals.

(iv) The approximant always passes through at least one of the data points.

Which are true?

A: Unanswered
B: (i) & (ii)
C: (iii) & (iv)
D: (i), (ii) & (iii)
E: (iii)
F: (ii) & (iii)
G: (i) & (iv)
H: All of the above

Question 5 The following data is given:

x	0	1	2
$f(x)$	0	0.476	1.619

Find the parabola that best fit the data in a least-square sense. What is approximately the value of the residual at $x = 1$ and $x = 2$ respectively?

A: Unanswered
B: $0.00, 0.00$
C: $0.00, 0.01$
D: $0.01, 0.00$
E: $0.01, 0.01$
F: $0.01, 0.43$
G: $0.43, 0.00$
H: $0.43, 0.01$

Question 6 The following distinct data is given:

x	x_0	x_1	x_2	x_3	x_4
$f(x)$	f_0	f_1	f_2	f_3	f_4

Given the following four sets of basis functions, which will give the best approximation of the data in the sense of minimizing the least-squares residual?

(i) $\varphi_1 = 1, \quad \varphi_2 = x$

(ii) $\varphi_1 = 1, \quad \varphi_2 = x, \quad \varphi_3 = x^2$

(iii) $\varphi_1 = 1, \quad \varphi_2 = x, \quad \varphi_3 = x^2, \quad \varphi_4 = x^3$

(iv) $\varphi_1 = 1, \quad \varphi_2 = x, \quad \varphi_3 = x^2, \quad \varphi_4 = x^3, \quad \varphi_5 = x^4$

A: Unanswered
B: (i)
C: (ii)
D: (iii)
E: (iv)
F: Not enough information to answer

Module 4: Numerical differentiation and integration

Question 7 The function $f(x)$ has everywhere positive second-derivatives - i.e. $f''(\xi) > 0$ for all $\xi \in \mathbb{R}$. What can be said about the truncation error

$$\varepsilon := Df - f'(x)$$

when using (a) forward-differences for D and (b) backward-differences for D, when approximating $f'(x)$? [Hint: Derive the trunction error using a 3-term Taylor-series.]

A: Unanswered
B: (a) $\varepsilon < 0$, (b) $\varepsilon > 0$
C: (a) $\varepsilon > 0$, (b) $\varepsilon < 0$
D: (a) $\varepsilon > 0$, (b) $\varepsilon > 0$
E: (a) $\varepsilon < 0$, (b) $\varepsilon < 0$
F: Depends on x.

Question 8 A high-order central-difference formula for the 1st-derivative of $f \in C^6([a, b])$ is:

$$f'(x_i) = \frac{f(x_i - 2h) - 8f(x_i - h) + 8f(x_i + h) - f(x_i + 2h)}{12h} + \epsilon$$

where ϵ is the truncation error. What is the value of the term in ϵ involving $f^{(5)}(x_i)$?

A: Unanswered
B: $-\frac{h^4 f^{(5)}(x_i)}{5}$
C: $-\frac{h^4 f^{(5)}(x_i)}{15}$
D: $-\frac{h^4 f^{(5)}(x_i)}{30}$
E: $-\frac{h^4 f^{(5)}(x_i)}{45}$

Question 9 Consider the integral:

$$I = \int_{-1}^{1} (e^x + 1)\,\mathrm{d}x$$

What is approximately the error ε made by approximating the integral with Simpson's rule? [Hint: Simpson's rule is the same as a 3-point closed Newton-Cotes rule.]

A: Unanswered
B: $\varepsilon = 0.012$
C: $\varepsilon = 0.080$
D: $\varepsilon = 0.125$
E: $\varepsilon = 0.235$
F: $\varepsilon = 0.535$
G: $\varepsilon = 1.235$

Question 10 **Fourier quadrature**. Until now we have considered quadrature rules that integrate polynomials of degree d exactly. Using the same principles derive a quadrature rule that integrates the functions $\phi = [1, \sin(x), \cos(x)]$ exactly on the interval $x \in [0, \pi]$. For nodes use $x = (0, \frac{\pi}{2}, \pi)$. What are the corresponding weights?

A: Unanswered
B: $w = (\frac{\pi}{2}, 0, \frac{\pi}{2})$
C: $w = (\frac{\pi}{2} - 1, 2, \frac{\pi}{2} - 1)$
D: $w = (-1, 2 + \pi, -1)$
E: $w = (\frac{\pi}{2} + 1, -2, \frac{\pi}{2} + 1)$
F: $w = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$
G: $w = (\frac{\pi}{3} - 1, \frac{\pi}{3} + 2, \frac{\pi}{3} - 1)$
H: $w = (0, \pi, 0)$