Applied Numerical Analysis – Quiz #1

Modules 1 and 2

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

Throughout we use $T_k[f, x_0](x)$ to denote the truncated Taylor series:

$$T_k[f, x_0](x) := \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i.$$

Question 1 Use a truncated Taylor-series to approximate the function $f(x) = \sin(x)$ near $x_0 = 0$, i.e. $f(x) \approx T_k[f, 0](x)$. What is the least value of k giving an error less than 10^{-3} at $x = \pi/4$?

A: Unanswered
B: 0
C: 1
D: 2
E: 3
F: 4
G: 5
H: 6

Question 2 The following algorithm is given in the reader: Assume an interval $[a, b]$, and a continuous function $f(x)$, such that $f(a)$ and $f(b)$ have opposite sign:

$a_0 \leftarrow a$; $b_0 \leftarrow b$
for $i = [0 : N]$ **do**
  $c_i \leftarrow \frac{1}{2}(a_i + b_i)$
  if $f(a_i) \cdot f(c_i) < 0$ **then**
    $a_{i+1} \leftarrow a_i$; $b_{i+1} \leftarrow c_i$
  else
    $a_{i+1} \leftarrow c_i$; $b_{i+1} \leftarrow b_i$
  end if
end for
return c_i

then c_N is guaranteed to be an approximation of a root of f.

Assume now that f is continuous but $f(a)$ and $f(b)$ have the same sign, and f has no root in $[a, b]$. What is the value of c_N?

A: Unanswered
B: Algorithm terminates due to bad input.
C: Algorithm finds a root.
D: $a + \frac{b-a}{2^{N+1}}$
E: $b - \frac{b-a}{2^{N+1}}$
F: $a + \frac{b-a}{2^N}$
G: $b - \frac{b-a}{2^N}$
H: b

Question 3 The function $f(x) = x^3 - 9$ has a root in the interval $x \in [2, 3]$. Using Newton's method starting with $x_0 = 2$, what is the *error* in the value of x_1?

A: Unanswered
B: Diverges after one step
C: 0.0
D: 3.24×10^{-1}
E: 3.24×10^{-2}
F: 3.24×10^{-3}
G: 3.24×10^{-4}
H: 3.24×10^{-5}

Question 4 The function $f(x) = \tan x - x$ has roots at $\tilde{x}_0 = 0$, and at $\tilde{x}_1$, a point near $3\pi/2$. These two roots are calculated using Newton's method, by starting with two initial guesses, one close to each root. What do you expect the rate of convergence Newton to be in these two cases, $\tilde{x}_0$ and $\tilde{x}_1$ respectively? [Hint: Consider the derivative of $f(x)$ at each root.]

A: Unanswered
B: Linear, Linear
C: Linear, Quadratic
D: Quadratic, Linear
E: Quadratic, Quadratic

Question 5 Approximate the function $f(x) = \cos(\pi x)$ by a polynomial p of degree 2 in the interval $x \in [0, 1]$. Interpolate f at $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$, using any desired basis. What is the exact error $\epsilon(x) = |f(x) - p(x)|$ at $x = \frac{1}{4}$?

A: Unanswered
B: $\frac{1}{\sqrt{2}} + \frac{1}{2}$
C: $\frac{1}{\sqrt{2}} - \frac{1}{2}$
D: $\frac{2}{\sqrt{2}} + 1$
E: $\frac{2}{\sqrt{2}} - 1$
F: $\frac{1}{2\sqrt{2}}$
G: $-\frac{1}{2\sqrt{2}}$

Question 6 In the previous question we approximated a wave with a polynomial. Now we approximate a polynomial with a Fourier series. Using the basis functions

$$\varphi_0(x) = 1, \quad \varphi_1(x) = \cos x, \quad \varphi_2(x) = \sin x,$$

interpolate $f(x) = x^2$ using uniformly-spaced nodes on the interval $[0, \pi]$. What is the derivative of the interpolant at $x = 0$?

A: Unanswered
B: 0
C: $\pi/4$
D: $-\pi/4$
E: $\pi^2/4$
F: $-\pi^2/4$
G: $\pi/2$
H: $-\pi/2$

Question 7 Given a basis $\varphi_j(x)$, $j = 0, \dots N$, and a sample grid x_i, $i = 0, \dots N$, the interpolation matrix is defined by

$$A_{ij} = \varphi_j(x_i).$$

You are given an 8×8 lower-triangular interpolation matrix (i.e. $A_{ij} = 0, \forall j > i$). What do you know about $\varphi_3(x)$?

A: Unanswered
B: $\varphi_3(x)$ matches the data at x_i, $i = 0, \dots, 2$
C: $\varphi_3(x)$ matches the data at x_i, $i = 4, \dots, 8$
D: $\varphi_3(x)$ has roots at x_i, $i = 0, \dots, 2$
E: $\varphi_3(x)$ has roots at x_i, $i = 4, \dots, 8$
F: $\varphi_3(x)$ has at least 4 roots
G: $\varphi_3(x)$ has at most 2 roots

Question 8 An interpolating polynomial passes through the points

x	3	5	x_2
y	20	12	30

The polynomial is of the form:

$$p(x) = a_0 + a_1(x - 3) + 2(x - 3)(x - 5).$$

What are the two possible values for x_2? [Hint: Deduce a_0 first, then a_1, and finally solve for x_2.]

A: Unanswered
B: $x_2 = 1, 9$
C: $x_2 = 2, 8$
D: $x_2 = 3, 7$
E: $x_2 = 4, 6$
F: $x_2 = 5, -5$
G: $x_2 = 6, -6$

Question 9 Consider the following two facts:

(i) The Weierstrass theorem states: For any $f \in C^0([a,b])$ and any $\epsilon > 0$, there exists a polynomial p, such that $\max_{\xi\in[a,b]} |f(\xi) - p(\xi)| < \epsilon$.

(ii) For a uniform grid with $N+1$ points, polynomial approximations of the Runge function $f_R(x) = \frac{1}{1+x^2}$ have large oscillations.

What is the explaination for this apparent contradiction? (An equivalent question is: Which of the following statements is true?)

A: Unanswered
B: The mentioned osciallations decrease in size as $N \to \infty$.
C: Weierstrass isn't always true; the Runge function is an exception.
D: $f_R \in C^\infty(\mathbb{R})$ (infinitely continuously-differentiable) so Weierstrass doesn't apply.
E: $f_R \in C^1(\mathbb{R})$ (once continuously-differentiable) so Weierstrass doesn't apply.
F: Weierstrass doesn't hold for polynomials that must intersect f at specified points.
G: Weierstrass only applies on an interval $[a,b]$, and approximations of $f_R(x)$ only osciallate if the entire real-line $\mathbb{R}$ is considered.

Question 10 In constrast to Weierstrass, Cauchy gives us an explicit expression for the error in an $N+1$ point polynomial interpolant p_N:

$$f(x) - p_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}\prod_{i=0}^{N}(x - x_i)$$

for some value of $\xi \in [x_0, x_N]$, that is usually unknown. However, ξ can be calculated if f is known exactly.

For the function $f(x) = \sin(\pi x)$, the grid $(x_0, x_1, x_2) = (0, \frac{1}{2}, 1)$, and the error at $x = \frac{1}{4}$, what is approximately the value of $\cos(\pi\xi)$?

A: Unanswered
B: 0.177
C: 0.277
D: 0.377
E: 0.477
F: 0.577
G: 0.677
H: 0.777