Applied Numerical Analysis – Quiz #3

Modules 5 and 6

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

Module 5: Numerical solution of ODEs

Throughout we assume the standard form of the ODE: $\mathbf{u}'(t) = f(\mathbf{u}(t))$.

Question 1 The ODE $u' = -\frac{5}{2}u$ with initial condition $u(0) = 2$ is solved numerically using (a) the forward-Euler method (explicit), and (b) the backward-Euler method (implicit). In both cases the stepsize $\Delta t = 0.6$. Of the two methods, which are stable?

A: Unanswered
B: Neither
C: (a) only
D: (b) only
E: Both

Question 2 A function $f(u)$ is Lipschitz continuous on an interval $[a, b]$ if: there exists some Lipschitz constant $0 < L < \infty$, such that for all u_1 and u_2 in $[a, b]$

$$|f(u_1) - f(u_2)| \le L|u_1 - u_2|.$$

The function $f(u) = u^2$ is Lipschitz continuous on $[0, 10]$. What is the minimum constant L needed in the definition above, for this function on this interval?

A: Unanswered
B: 1/5
C: 1/2
D: 1
E: 2
F: 5
G: 10
H: 20

Question 3 Which of the following statements are true?

i. A scalar ODE of the form $y^{(M)} + y^{(M-1)} + \cdots + y' = F(y)$, can always be transformed into an equivalent system of ODEs $\mathbf{u}' = f(\mathbf{u})$, with $\mathbf{u}$ a vector of size M.

ii. Discretization errors typically decrease with decreasing step size.

iii. Floating-point rounding errors typically increase with decreasing step size.

iv. If the local discretization error of a scheme is $\mathcal{O}(\Delta t^p)$, its global truncation error is always $\mathcal{O}(\Delta t^{p-1})$.

A: Unanwered
B: None
C: i, ii
D: i, ii, iii
E: i, iii, iv
F: ii, iv
G: ii, iii, iv
H: All

Question 4 Consider the ODE $u' = -cu$ with initial condition $u(0) = 1$ and constant $c > 0$. Using backward Euler:

$$u_{i+1} = u_i + \Delta t f(u_{i+1})$$

and $\Delta t = \frac{1}{2}$, what is the approximation of $u(4)$?

A: Unanswered
B: $(1 + c/2)^{-8}$
C: $(1 + c/2)^8$
D: $(1 - c/2)^{-8}$
E: $(1 - c/2)^8$
F: $(c/2)^{-8}$
G: $(c/2)^8$

Question 5 Consider the scheme:

$$u_{i+1} = u_i + \frac{3}{2}\Delta t f(u_i) + \frac{1}{2}\Delta t f(u_{i-1}),$$

where u_i and u_{i-1} are known, and u_{i+1} is unknown. What is the local-truncation error of this scheme? [Hint: Taylor expand u_{i-1}, then $f(u_{i-1})$.]

A: Unanswered
B: Δt^0
C: Δt^1
D: Δt^2
E: Δt^3

Module 6: Numerical optimization

Question 6 Consider an objective function $J(x)$ for $x \in \mathbb{R}$. Near some unspecified x_0, J can be approximated by $J(x_0 + h) = 4 - 2h^2 + \mathcal{O}(h^3)$ with $|h| << 1$. What can be said about J at x_0? It is a:

A: Unanswered
B: Local minimum
C: Local maximum
D: Global minimum
E: Global maximum
F: Zero point (root)
G: Inflection point
H: None of the above

Question 7 Consider minimizing $f(x)$. Which of the following is *false*?

i. The golden-section search is guaranteed to reduce the width interval by a factor of $(1+\sqrt{5})/2$ at each successful iteration

ii. Newton minimizing $f(x)$ is equivalent to finding a root of $f'(x)$

iii. If Newton converges to a minimum it will converge quadratically

iv. Steepest descent is guaranteed to converge to the global minimum

v. The Nelder-Mead simplex method requires $d+1$ evaluations of f to generate the initial simplex if x is d-dimensional

A: Unanswered
B: i
C: ii
D: iii
E: iv
F: v
G: All are true
H: All are false

Question 8 A rectangular cold storage box with *square* base of edge length l meters, height h meters and perfectly insulated top has a total volume of $10\,\mathrm{m}^3$. All other sides are uninsulated. The goal is to find l such that heat loss is minimized.
Apply Newton's method for optimization to minimize a suitable objective function (assume heat loss is proportional to surface area). Perform the update in terms of l, starting with an initial estimate $l_0 = 1.0$. What is l_1 to two decimal places?

A: Unanswered
B: 0.46
C: 1.46
D: 1.19
E: 1.23
F: 0.54
G: 2.46
H: 1.54

Question 9 Consider $\tilde{x}$ the unique minimum of the function $f(x) = 7x - ln(x)$. Using Newton's method for minimization, and starting from $x_0 = 0.1$, what is the error in the approximation after one iteration?

A: Unanswered
B: 0.129
C: 0.0129
D: 0.00129
E: 0.000129
F: 0.0000129

Question 10 Steepest descent for minimizing $f(x, y)$ in 2-dimensions requires the gradient: $\nabla f(x_i, y_i)$ on iteration i. Assuming the function $\nabla f(x, y)$ is not known, it can be approximated at (x_i, y_i) by a difference rule (e.g. forward/backward/central differences). What is the *minimum* number of samples of $f(\cdot)$ required to approximate the gradient at (x_i, y_i) in this way?

A: Unanswered
B: 1
C: 2
D: 3
E: 4
F: 5