Applied Numerical Analysis – Quiz #2

Modules 3 and 4

Name: ______________________________ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

Module 3: Advanced interpolation

Question 1 Let $\phi(x)$ be a cubic spline on $x \in [a, b]$. What is the *most* that can be said about $\phi'(x)$ in general?

A: Unanswered
B: It is discontunous
C: It is continuous, i.e. $\phi' \in C^0([a, b])$
D: It is once continuously differentiable, i.e. $\phi' \in C^1([a, b])$
E: It is twice continuously differentiable, i.e. $\phi' \in C^2([a, b])$
F: It is infinitely differentiable, i.e. $\phi' \in C^\infty([a, b])$

Question 2 Consider the quadratic function $f(x) = 1 - x^2$ on the interval $x \in [-1, 1]$. Samples of $f(x)$ are taken at $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. Which of the following methods, when applied to these samples, will give an exact interpolation of $f(x)$ on the interval?

a. Polynomial interpolation

b. Linear spline interpolation

c. Cubic spline interpolation with natural BCs ($\phi''(x_0) = \phi''(x_2) = 0$)

d. Cubic spline interpolation with "clamped" BCs ($\phi''(x_0) = f''(x_0)$ and $\phi''(x_2) = f''(x_2)$)

A: Unanswered	C: All	E: a, c, d	G: a, d
B: None	D: a	F: a, c	H: b

Question 3 Again a cubic spline is used to approximate $f(x) = 1 - x^2$ on the interval $x \in [-1, 1]$ using samples at $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. The two branches of the spline are:

$$\phi(x) = \begin{cases} S_0(x) = a_0(x+1)^3 + b_0(x+1)^2 + c_0(x+1) + d_0 & \text{if } -1 \le x < 0, \\ S_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1 & \text{if } 0 \le x \le 1 \end{cases}$$

Natural BCs are used, i.e. $\phi''(x_0) = \phi''(x_2) = 0$. Further it is known that $a_0 = -\frac{1}{2}$ and $b_0 = 0$. What is the value of a_1?

A: Unanswered	C: -1	E: 0	G: 1
B: $-\frac{3}{2}$	D: $-\frac{1}{2}$	F: $\frac{1}{2}$	H: $\frac{3}{2}$

Question 4 Consider tensor-product polynomial interpolation in 2d, with grid consisting of the rectangular lattice with nodes $X_{ij} = (x_i, y_j)$, with $\mathbf{x} = (0, 1, 2, 3, 4)$ and $\mathbf{y} = (0, \frac{1}{2}, 1)$. Which of the following monomials are represented exactly by the interpolation?

a. xy b. x^4y^2 c. x^2y^2 d. x^2

A: Unanswered	C: None	E: d	G: b
B: All	D: a, b, c	F: a, c, d	H: a, b

Question 5 In radial-basis function interpolation a different radial function can be used at each node. Consider interpolation in 1d on the 3-node grid $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. Each node has a corresponding basis function:

$$\varphi_0(x) = \frac{1}{10}|x+1|, \qquad \varphi_1(x) = |x|, \qquad \varphi_2(x) = 10|x-1|.$$

By setting up the interpolation conditions for the interpolant

$$\phi(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + a_2\varphi_2(x),$$

determine the value of a_0, given that $f_0 = 1$, $f_1 = 0$, and $f_2 = 2$, and that $a_2 = -\frac{1}{40}$.

A: Unanswered
B: $-\frac{5}{2}$
C: $-\frac{3}{2}$
D: $-\frac{1}{2}$
E: 0
F: $\frac{1}{2}$
G: $\frac{3}{2}$
H: $\frac{5}{2}$

Module 4: Numerical differentiation and Integration

Question 6 When deriving cubic splines we developed the system of equations:

$$\frac{h}{6}M_{i-1} + \frac{2h}{3}M_i + \frac{h}{6}M_{i+1} = \frac{f_{i+1}-f_i}{h} - \frac{f_i - f_{i-1}}{h},$$

for the unknowns M_i. The right-hand side of this equation (for a fixed i) looks a bit like a difference formula for some derivative of $f(x)$ based on three samples $f(x_i - h)$, $f(x_i)$ and $f(x_i + h)$. What is it approximating? [Hint: Taylor...]

A: Unanswered
B: $f'(x_i)$
C: $hf'(x_i)$
D: $2hf'(x_i)$
E: $f''(x_i)$
F: $hf''(x_i)$
G: $2hf''(x_i)$
H: $6h^2 f'''(x_i)$

Question 7 Difference rules can be found by differentiating a polynomial interpolant. Consider a 3-node grid $(x_0, x_1, x_2) = (-h, 0, h)$, with corresponding function values (f_0, f_1, f_2). Using a Lagrange basis, first find a quadratic interpolant, and then differentiate it. Obtain an approximation of $f'(h/3)$, of the form:

$$f\left(\frac{h}{3}\right) \approx \frac{1}{h}(a_0 f_0 + a_1 f_1 + a_2 f_2),$$

Which of the following is the resulting approximation of $f'(x)$ at $x = \frac{h}{3}$?

A: Unanswered
B: $\frac{1}{h}(-f_0 + f_1)$
C: $\frac{1}{2h}(-f_0 + f_2)$
D: $\frac{1}{3h}(-2f_0 - f_1 + 3f_2)$
E: $\frac{1}{6h}(-f_0 - 2f_1 + 3f_2)$
F: $\frac{1}{6h}(-f_0 - 4f_1 + 5f_2)$
G: $\frac{1}{6h}(-4f_0 - f_1 + 5f_2)$

Question 8 Simpson's rule defined on the interval $[-1, 1]$ is:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \simeq \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)$$

The same rule on a general interval $[a, b]$ is:

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{i=0}^{2} w_i f(x_i).$$

Determine the weights w_1 and w_2, where $h := b - a$.

A: Unanswered
B: $w_1 = 1$ and $w_2 = 1$
C: $w_1 = \frac{h}{6}$ and $w_2 = \frac{h}{6}$
D: $w_1 = \frac{2h}{3}$ and $w_2 = \frac{h}{6}$
E: $w_1 = \frac{h}{6}$ and $w_2 = \frac{2h}{3}$
F: $w_1 = \frac{4h}{3}$ and $w_2 = \frac{h}{3}$
G: $w_1 = \frac{h}{3}$ and $w_2 = \frac{4h}{3}$
H: $w_1 = h$ and $w_2 = 2h$

Question 9 For a function of two variables $f(x, y)$, we would like a finite-difference approximation of $\frac{\partial^2 f}{\partial x \partial y}$ at (x_0, y_0). This can be achieved by using forward-differences in x:

$$\frac{\partial f}{\partial x}(x_0, y_0) \simeq \phi(x_0, y_0) := \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

and then applying forward-differences in y to the function ϕ. Using the notation

$$f_{00} := f(x_0, y_0), \quad f_{10} := f(x_0 + \Delta x, y_0), \quad f_{01} := f(x_0, y_0 + \Delta y), \quad f_{11} := f(x_0 + \Delta x, y_0 + \Delta y)$$

What is the resulting expression for $\frac{\partial^2 f}{\partial x \partial y}$?

A: Unanswered
B: $(f_{11} - f_{00})/(\Delta x \Delta y)$
C: $(f_{01} - f_{10})/(\Delta x \Delta y)$
D: $(f_{11} - f_{01} - f_{10} + f_{00})/(\Delta x \Delta y)$
E: $(f_{11} - f_{01} + f_{10} - f_{00})/(\Delta x \Delta y)$
F: $(f_{11} + f_{01} - f_{10} - f_{00})/(\Delta x \Delta y)$
G: $(f_{10} + f_{01} - 2f_{00})/(\Delta x \Delta y)$

Question 10 From polynomial interpolation we know the *interpolation* error (Cauchy). For $f(x)$ approximated by an $N+1$ point polynomial interpolant p_N, the error at a point x is:

$$E(f; x) := |f(x) - p_N(x)| = \frac{|f^{(N+1)}(\xi)|}{(N+1)!} |\omega_{N+1}(x)|,$$

where the nodal polynomial

$$\omega_{N+1}(x) := \prod_{i=0}^{N} (x - x_i).$$

Using this formula, and assuming $f^{(N+1)}(\xi) = 1$ (for all x), which of the following is an estimate for the error in Simpson's rule for an *integral* on the interval $[-h, h]$? (Note: Simpson's rule has nodes at $\mathbf{x} = (-h, 0, h)$.)

A: Unanswered
B: 0
C: $h^3/2$
D: $h^3/12$
E: $h^3/48$
F: $h^4/2$
G: $h^4/12$
H: $h^4/48$