Applied Numerical Analysis – Quiz #1

Modules 1 and 2

Name: ____ Student number: ______________

DO NOT OPEN UNTIL ASKED

Instructions:

- Make sure you have a machine-readable answer form.
- Write your name and student number in the spaces above, **and** on the answer form.
- Fill in the answer form **neatly** to avoid risk of incorrect marking.
- Fill in the version number of your quiz (see bottom right, A-D) on the answer form.
- Use only pencil on the answer form, and correct with a rubber.
- This quiz requires a calculator.
- Each question has exactly one correct answer.
- Correct answers will receive 5 points, unanswered questions 1 point, incorrect answers 0 points.
- This quiz has **10 questions** and **4 pages** in total.

Module 1: Taylor, Root-finding, Floating-point

Question 1 One way to approximate integrals is by first approximating the integrand by a truncated Taylor-series, and then integrating the series by hand exactly (it is just a polynomial). Using this method, approximate

$$\int_0^\pi \sin x \,\mathrm{d}x$$

by approximating $\sin x$ with a truncated Taylor series about $x_0 = \pi/2$, up to terms including x^2, and then integrating the series. What is the value of the approximate integral?

A: Unanswered
B: $\frac{2\pi}{3}$
C: $\frac{4\pi}{5}$
D: $\pi - \frac{\pi^3}{24}$
E: $\pi - \frac{\pi^2}{8}$
F: π
G: 2

Question 2 The small-angle approximation for trigonometric functions is based on a Taylor expansion about $x = 0$, up to quadratic terms. For which angle does approximation of $\sin(x)$ have a relative error exceeding approximately 1.0%? [Note: Relative error: $\frac{f_{true} - f_{approx}}{f_{true}} \times 100\%$]

A: Unanswered
B: $7\pi/180$
C: $14\pi/180$
D: $21\pi/180$
E: $\pi/2$
F: π

Question 3 Using the property of fixed-point iterations, that they are guaranteed to converge if $|\varphi'(x)| < 1$ for $x \in [\tilde{x}, x_0]$, and noting that Newton is a fixed-point iteration with

$$\varphi(x) = x - \frac{f(x)}{f'(x)},$$

in what starting interval is Newton's iteration guaranteed to converge to the exact root $\tilde{x} = 0$ of $f(x) = \sin x = 0$?

A: Unanswered
B: $x_0 = (-\frac{\pi}{4}, \frac{\pi}{4})$
C: $x_0 = (-\frac{\pi}{2}, \frac{\pi}{2})$
D: $x_0 = (-\pi, \pi)$
E: $x_0 = (-2\pi, 2\pi)$
F: Newton never converges to 0.

Question 4 Consider an (unknown) fixed-point iteration $x_{n+1} = \varphi(x_n)$ with

$$|\varphi'(x)| < \frac{1}{4} \quad \text{for} \quad x \in (-\infty, \infty).$$

Let the error of the initial guess $\epsilon_0 = 1$. At most how many iterations are required such that the error on iteration N satisfies $\epsilon_N < 2^{-32}$?

A: Unanswered
B: 4
C: 8
D: 16
E: 32
F: The iteration diverges.
G: The error is never that small.

Question 5 Consider again the problem of approximating the exact root $\tilde{x} = 0$ of $f(x) = \sin x = 0$, and the three fixed-point iterations, $x_{n+1} = \varphi_i(x_n)$ with:

$$\varphi_1 = x + \sin x$$
$$\varphi_2 = x - \sin x$$
$$\varphi_3 = \sqrt{x^2 + \sin x}.$$

Consider a starting guess x_0 extremely close (but not equal to) 0. Which of φ_1, φ_2, φ_3 are:

(a) Valid fixed-point iterations for the stated problem?, and

(b) Converge to 0 for the given starting point?

A: Unanswered
B: (a) all, (b) all
C: (a) all, (b) 1,2
D: (a) all, (b) 2
E: (a) 1,2 (b) none
F: (a) 1,2, (b) 1,2
G: (a) 1,2, (b) 2
H: (a) 1, (b) 1

Question 6 Consider Newton's iteration applied to the problem $f(x) = x^{2m} = 0$, where m is a natural number and $m \geq 1$. These are curves that only touch the axis. From the analysis of fixed-point iterations we know that the error behaves like:

$$e_{N+1} = \varphi'(\tilde{x})e_N + \frac{1}{2}\varphi''(\tilde{x})e_N^2 + \mathcal{O}(e_N^3)$$
$$e_N = x_N - \tilde{x}.$$

Given this, what can be said about the convergence of Newton for these problems?

A: Unanswered
B: Convergence is quadratic
C: Convergence is linear with rate $\frac{2m-1}{2m}$
D: Convergence is linear with rate $\frac{2}{m}$
E: Newton doesn't converge

Module 2: Polynomial Interpolation and Regression

Question 7 Suppose an approximation table is to be prepared for the function $f(x) = \sqrt{x}$ on the interval $[a, b] = [1, 2]$ with samples at equal spacing h. On each set of 3 consective points a polynomial of degree 2 is constructed. Approximately what value of h ensures the approximation error $\varepsilon \leq 5 \times 10^{-8}$?
Use the fact that for equidistant nodes, and polynomial degree N the approximation error is

$$\varepsilon_N(x) \leq \max_{x\in[a,b]} |f^{N+1}(x)| \frac{|h^{N+1}|}{4(N+1)}.$$

A: Unanswered
B: $h \approx 0.1$
C: $h \approx 0.01$
D: $h \approx 0.001$
E: $h \approx 0.0001$
F: $h \approx 0.00001$

Question 8 Consider an unknown function $f(x)$ which is sampled $N+1$ times at distinct coordinates x_i. *For these samples* we are able to find interpolating functions $\phi_M(x)$ defined as

$$\phi_M(x) = \sum_{i=0}^{M} a_i x^i,$$

and this is possible for $M = 1$, and $M = 2$ and $\ldots$ and $M = N$. Which **one** of the following options is true?

A: Unanswered
B: $f(x)$ is linear
C: $f(x)$ is quadratic
D: $a_0 = 0$ in all $\phi_M(x)$
E: All $\phi_M(x)$ are identical

Question 9 A data-set (x_i, f_i) containing $N+1$ points with x_i all distinct, is approximated in 3 ways:

a) Polynomial interpolation, giving $p(x)$

b) Least-squares regression with, giving $q(x)$

c) A modified regression, minimizing the 4th-power of the defects, i.e. $\min \sum_{i=0}^{N}(f_i - r(x_i))^4$, giving $r(x)$

In the two regression cases, $q(x)$ and $r(x)$ are degree M polynomials with $M < N$. Which of the following is true *in general*?

A: Unanswered
B: $p = q = r$
C: $p = q \neq r$
D: $p \neq q \neq r$
E: $p \neq q = r$

Question 10 Approximate $f(x) = \sin(\frac{1}{2}\pi x)$ with a quadratic polynomial and sample points at $x = (0, 0.5, 1)$. Use your prefered choice of basis. What is the non-trivial root of the approximation to 2-significant-figures?

A: Unanswered
B: 2.0
C: 2.1
D: 2.2
E: 2.3
F: 2.4