Applied Numerical Analysis – Homework # 5

Numerical Integration and Differentiation

Note: Throughout this quiz we consider the standard form of the ODE to be:

$$y'(x) = f(x, y(x)).$$

Preliminaries of ODEs

Question 1 How may the following ODE be classified?

$$\begin{aligned} y''' - 2y'' + y &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \\ y'(1) &= 0 \\ t &\in [0, 1] \end{aligned}$$

A: 2nd-order linear BVP
B: 2nd-order linear IVP
C: 2nd-order non-linear BVP
D: 2nd-order non-linear IVP
E: 3rd-order linear BVP
F: 3rd-order linear IVP
G: 3rd-order non-linear BVP
H: 3rd-order non-linear IVP

Question 2 Lipschitz continuity is an property of a function that is stronger than continuity, but weaker than differentiability, and is important in establishing the uniqueness of solutions of ODEs. A function $f(y)$ is Lipschitz continuous on $[a, b]$ if there exists an $L > 0$ such that

$$|f(y_1) - f(y_2)| \leq L|y_1 - y_2|,$$

for all y_1, y_2 in $[a, b]$. Which of the following functions *are* continuous, but *not* Lipschitz continuous in y where $y \in [-1, 1]$? (Note: see also lecture notes on uniqueness of solutions of ODEs.)

(i) $f(y) = |y|$
(ii) $f(y) = |y|^{\frac{1}{3}}$
(iii) $f(y) = y^{\frac{1}{3}}$
(iv) $f(y) = |\frac{1}{y}|$
(v) $f(y) = \tan y$
(vi) $f(y) = H(y)$

($H(y)$ is the Heaviside function, $H(y) = 1$ for $y > 0$ and $H(y) = 0$ otherwise.)

A: (i), (ii), (iv)
B: (i), (iii), (v)
C: (i), (ii)
D: (ii), (iii), (v)
E: (ii), (iii)
F: (iv), (v)
G: None.
H: All.

Question 3 Let $f(y)$ be Lipschitz continuous on an interval. What does this imply about solutions of the initial-value problem (IVP) $y' = f(y)$ in that interval? (Cauchy-Lipschitz theorem)

A: If f is also continuous, a unique solution exists.
B: A solution exists, but may not be unique.
C: A unique solution exists.
D: Multiple solutions may exist.
E: A solution is unique, but doesn't exist.
F: Nothing can be said about the solution.

Question 4 Consider the linear ODE $y' = 2\lambda y$, where λ is a constant. For what values of λ is the solution y stable? [Stable: small perturbations of the initial conditions produce small perturbations in the solution.]

A: $\lambda > \frac{1}{2}$
B: $\lambda < \frac{1}{2}$
C: $\lambda < 2$
D: $\lambda > 2$
E: $\lambda > 1$
F: $\lambda < 1$
G: $\lambda < 0$
H: $\lambda > 0$

Question 5 Consider the ODE:

$$y' + 2y = -e^y.$$

Which of the options below forms a suitable linearisation (about $y = y_n$) of this ODE?

A: $y' + 2y = -y_n e^{y_n}$
B: $y' + 2y = -e$
C: $y' + (2 + e^{y_n})y = e^{y_n}(y_n - 1)$
D: $e^{y_n} + e^{y_n}(y - y_n) = 0$
E: $y' + 2y = -e^y - he^y$
F: $y' + (y_n - 1)\,y = e^{y_n}(2 + e^{y_n})$

Numerics of ODEs

Question 6 Which of the answers below describes best an explicit time integration method?

A: New values x_{i+1} and y_{i+1} are computed only with the new (unknown) values.
B: Each increment is computed in terms of previous values.
C: The solution at any point can be explicitly solved for in terms of the initial conditions.
D: The global error is always $O(\Delta t^2)$.
E: They are inefficient.
F: The maximum stable time-step tends to be larger than for implicit methods.
G: The global error is always zero.

Question 7 If the *local* truncation error of a certain time integration scheme is $O(h^5)$, what can you say about the expected order of the *global* truncation error at a fixed time T?

A: $O(h^3)$
B: $O(h^4)$
C: $O(h^5)$
D: $O(h^6)$
E: This cannot be determined without knowing the scheme considered.

Question 8 Compute the order of magnitude of the local truncation error of the following time integration scheme:

$$y_{n+1} = y_{n-1} + 2hf(y_n)$$

A: 0
B: $O(h)$
C: $O(h^2)$
D: $O(h^3)$
E: $O(h^4)$
F: ∞

Question 9 Compute the leading order term of the local truncation error of the following time integration scheme for the standard form of an ODE:

$$y_{n+1} = \frac{6y_n - y_{n-1} - y_{n-2} + hf(y_n)}{4}$$

A: $\frac{9}{8}y''(x_n)h$
B: $\frac{9}{8}y''(x_n)h^2$
C: $9y''(x_n)h$
D: $9y''(x_n)h^2$

Question 10 Consider the initial-value problem:

$$y' = -3y + 1, \qquad y(0) = 2.$$

We want to apply a forward Euler scheme, with a time step $\Delta t = 0.001$. Which of the following implementations is correct?

A: $y_{n+1} = y_n - 0.003y_n + 2$
B: $y_{n+1} = y_n - 0.003y_n + 0.002$
C: $y_{n+1} = y_n - 0.003y_n + 0.001$
D: $y_{n+1} = y_n - 0.001y_n + 2$
E: $y_{n+1} = y_n - 0.001y_n + 0.002$
F: $y_{n+1} = y_n - 0.001y_n + 1$
G: $y_{n+1} = y_n - 0.001y_n + 0.001$
H: $y_{n+1} = y_n + 0.003y_n$

Question 11 Consider the initial-value problem:

$$y' = 2y - 5, \qquad y(0) = 0.$$

We want to apply a forward Euler scheme, with a time step $\Delta t = 0.002$. Which of the following implementations is correct?

A: $y_{n+1} = 1.002y_n - 0.01$
B: $y_{n+1} = 1.010y_n - 0.01$
C: $y_{n+1} = 1.004y_n - 0.01$
D: $y_{n+1} = 1.002y_n - 0.05$
E: $y_{n+1} = 1.010y_n - 0.05$
F: $y_{n+1} = 1.004y_n - 0.05$
G: $y_{n+1} = 1.002y_n$
H: $y_{n+1} = 1.010y_n$

Question 12 Consider the initial-value problem:

$$y' = -0.1y, \qquad y(0) = 1$$

We want to use a backward Euler scheme with a time step $\Delta t = 0.01$. Which of the following implementations is correct?

A: $y_{n+1} = \frac{y_n}{1.001} + 1$
B: $y_{n+1} = \frac{y_n}{1.001} - 0.1$
C: $y_{n+1} = \frac{y_n}{1.001}$
D: $y_{n+1} = \frac{y_n}{1.01} + 1$
E: $y_{n+1} = \frac{y_n}{1.01} - 0.1$
F: $y_{n+1} = \frac{y_n}{1.01}$
G: $y_{n+1} = 1.01y_n - 0.1$
H: $y_{n+1} = 1.001y_n$

Question 13 Solve the initial-value problem

$$y' = -100y, \qquad y(0) = 1,$$

using the *forward* Euler scheme

$$y_{n+1} = y_n + \Delta t f(y_n).$$

Use a time-step of $\Delta t = 0.001$, and perform 3 steps. What is your approximation of y at $t = 0.003$?

A: 1.0000
B: 0.9091
C: 0.7200
D: 0.7290
E: 0.2451
F: 0.5129
G: 0.9811
H: 0.1434

Question 14 Solve the initial-value problem

$$y' = -100y, \qquad y(0) = 1,$$

using the *backward* Euler scheme

$$y_{n+1} = y_n + \Delta t f(y_{n+1}).$$

Use a time-step of $\Delta t = 0.001$, and perform 3 steps. What is your approximation of y at $t = 0.003$?

A: 1.0000
B: 0.9091
C: 0.8264
D: 0.7513
E: 0.6630
F: 0.5129
G: 0.2892
H: 0.1434

Question 15 An unspecified numerical scheme, with a stability region given by

$$\frac{1}{2}|z+2| < 1$$

($z = hf_y$) is used to solve the ODE:

$$\mathbf{y}' = Ay,$$

where A is a 2×2 matrix with a complex-conjugate pair of eigenvalues:

$$\lambda_1 = a(1+i), \qquad \lambda_2 = a(1-i).$$

where $i = \sqrt{-1}$. The stepsize $h = 0.01$ is choosen. For what values of a is the scheme stable?

A: $0 > a > -400$
B: $0 < a < 400$
C: $0 > a > -200$
D: $0 < a < 200$
E: $0 > a > -100$
F: $0 < a < 100$
G: Never stable.
H: Always stable.

Question 16 Consider the following implicit scheme:

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(y_{n+1}) + f(y_n)\right].$$

By defining $\Delta y_n := y_{n+1} - y_n$, and linearizing the term $f(y_{n+1})$ about y_n, one can obtain an explicit scheme which is an approximation to this – with approximation error $O(\Delta t^3)$. Which of the following is that explicit scheme?

A: $[1 - \frac{1}{2}\Delta t f_y]\Delta y_n = \Delta t f(y_n)$
B: $[1 + \frac{1}{2}\Delta t f_y]\Delta y_n = \Delta t f(y_n)$
C: $[1 - \Delta t f_y]\Delta y_n = \Delta t f(y_n)$
D: $[1 + \Delta t f_y]\Delta y_n = \Delta t f(y_n)$

Question 17 Consider a Runge-Kutta time integration scheme with 2-stages:

$$y_{n+1} = y_n + \frac{1}{2}k_1 + \frac{1}{2}k_2,$$
$$k_1 = \Delta t f(y_n),$$
$$k_2 = \Delta t f(y_n + \beta k_1),$$

where $f(y) = cy$, with c a complex number. Let $z = c\Delta t$. For what values of z is the scheme stable?

A: $|1 + \beta z| < 1$
B: $|1 + z + \frac{1}{2}\beta z^2| < 1$
C: $|1 + z + \frac{1}{2}\beta z^2 + \frac{1}{12}z^3| < 1$
D: $|1 + z + \frac{1}{2}\beta z^2 + \frac{1}{12}z^3 + \frac{1}{64}z^4| < 1$
E: $|1 + z + \beta z^2| < 1$
F: $|1 + 2\beta z| < 1$
G: $|1 + z + \beta z^2 + \frac{1}{24}z^3| < 1$
H: $|1 + z + \beta z^2 + \frac{1}{24}z^3 + \frac{1}{96}z^4| < 1$

Question 18 We consider the motion of an aerofoil in 2d, which is allowed to oscillate vertically and rotate about it's elastic axis, and is forced by aerodynamics. This may be modelled as a coupled system of 2 ODEs (c.f. Aeroelasticity). If the vertical position of the aerofoil is h and it's rotation θ, then the motion is described by

$$\begin{aligned} m\ddot{h} + K_h h &= -L, \\ I_\theta \ddot{\theta} + K_\theta \theta &= M, \end{aligned}$$

where L is the lift force, M is the pitching moment, and the remaining constants represent the structure. We wish to rewrite this as a system of first-order ODEs in the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x} + \mathbf{b}.$$

Which of the following contains the correct expression for A and $\mathbf{b}$?

A:

$$\mathbf{x} = \begin{Bmatrix} h \\ \dot{h} \\ \theta \\ \dot{\theta} \end{Bmatrix}, \; A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_h}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{K_\theta}{I_\theta} & 0 \end{bmatrix}, \; \mathbf{b} = \begin{Bmatrix} 0 \\ -\frac{L}{m} \\ 0 \\ \frac{M_y}{I_\theta} \end{Bmatrix}$$

B:

$$\mathbf{x} = \begin{Bmatrix} h \\ \dot{h} \\ \theta \\ \dot{\theta} \end{Bmatrix}, \; A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_h}{m} & -\frac{L}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{K_\theta}{I_\theta} + \frac{M_y}{I_\theta} & 0 \end{bmatrix}, \; \mathbf{b} = \mathbf{0}$$

C:

$$\mathbf{x} = \begin{Bmatrix} h \\ \dot{h} \\ \theta \\ \dot{\theta} \end{Bmatrix}, \; A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_h}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{K_\theta}{I_\theta} & 0 \end{bmatrix}, \; \mathbf{b} = \mathbf{0}$$

Question 19 For the (continuous) aeroelastic system in Question 18 to be stable we require that both h and θ remain finite as $t \to \infty$. Under what conditions on the matrix A will this be true?

A: A is postitive definite.
B: $\det(A) > 0$
C: $\det(A) = 0$
D: $\det(A) < 0$
E: All eigenvalues of A are real.
F: $\mathrm{Re}(\lambda) > 0$, for λ any eigenvalue of A.
G: $\mathrm{Re}(\lambda) < 0$, for λ any eigenvalue of A.

Question 20 Apply two iterations of the Runge-Kutta scheme of Question 17 with $\beta = 1$ to the system of Question 18. Use the initial conditions $h = 1$, $\dot{h} = 0$, $\theta = 0$, $\dot{\theta} = 1$ at $t = 0$, and a timestep $\Delta t = 1/100$. Further assume all constants are equal to 1. What is the value of h at time $t = 2/100$?

A: 0.99960   B: 0.79990   C: 0.59990   D: 0.39990

Quiz 2013 – 60 mins

Question 21 Consider the following attempt to construct a scheme based on quadrature. We rewrite the scalar ODE $y' = f(y)$ on the interval $[t_n, t_{n+1}]$ as:

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y(t))\,\mathrm{d}t, \tag{1}$$

and then approximate the integral with a closed Newton-Cotes rule with $N + 1 > 2$ points:

$$\int_{t_n}^{t_{n+1}} f(y(t))\,\mathrm{d}t \approx \sum_{i=0}^{N} w_i f\left(y\left(t_n + \Delta t \frac{i}{N}\right)\right). \tag{2}$$

with correctly chosen weights w_i. The method as described is not a valid scheme. Which of the following best describes the problem?

A: Unanswered
B: The standard ODE can not be rewritten as equation (1).

C: Quadrature applies to functions like $f(t)$, not like $f(y(t))$.
D: The approximation in equation (2) is poor.
E: The update to y_n (the integral) is not multiplied by Δt.
F: The values of y at the quadrature nodes are unknown.
G: The method as described is unstable.
H: The method has local truncation error of $\mathcal{O}(1)$.

Question 22 Consider the following three scalar initial-value problems:

(i) $y' = \sqrt{|y|}$

(ii) $y' = y$

(iii) $y' = y^{\frac{2}{3}}$

Where in each case the initial condition $y(0) = y_0 \in \mathbb{R}$ can be any real number. Which of the above initial-value problems are guaranteed to be uniquely solvable according to the Cauchy-Lipschitz Theorem?

A: Unanswered
B: (i)
C: (ii)
D: (iii)
E: (i) and (ii)
F: (i) and (iii)
G: All
H: None

Question 23 Which of the following numerical schemes are *implicit*?

(i) $y_{n+1} = y_n + \Delta t f(y_n)$

(ii) $y_{n+1} = y_n + \Delta t f(y_{n+1})$

(iii) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(\hat{y})\right), \quad \hat{y} = y_n + \Delta t f(y_n)$

(iv) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(y_{n+1})\right)$

(v) $y_{n+1} = y_n + \frac{\Delta t}{2}\left(f(y_n) + f(y_{n-1})\right)$

A: Unanswered
B: (i), (iii)
C: (ii), (iv)
D: (ii), (iii), (iv)
E: (ii), (iv), (v)
F: (ii), (iii), (iv), (v)
G: (iii), (iv), (v)
H: (iv), (v)

Question 24 A numerical scheme with a local truncation error of $\mathcal{O}(\Delta t^4)$ is used to solve $y' = f(y)$, $y(0) = 1$ and evaluate $y(T)$ - the solution at a fixed time $T > 0$. The timestep used is very small: $\Delta t \ll 1$. The error in $y(T)$ is estimated as ε.

The same scheme is applied again with a timestep of $\frac{\Delta t}{2}$, starting from the initial condition. What is approximately the new error in $y(T)$? [Note: Assume exact arithmetic is used.]

A: Unanswered
B: 2.7×10^{-4}
C: 2ε
D: ε
E: $\varepsilon/2$
F: $\varepsilon/4$
G: $\varepsilon/8$
H: $\varepsilon/16$

Question 25 Consider the ODE $y'' - 0.5y' + y = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$. Compute y' after one iteration of the forward Euler scheme with $\Delta t = 0.1$. [Hint: First write this ODE as a system of first order ODEs.]

A: Unanswered
B: 0.9434
C: 0.9448
D: 1
E: 1.0417
F: 1.0439
G: 1.0481
H: 1.05

Question 26 For the initial value problem $y' = cy, y(t_0) = y_0$, and $c \in \mathbb{C}$ a complex number, we consider the method:

$$y_{n+1} = y_n + \Delta t[\alpha f_n + (1-\alpha) f_{n-1}],$$

where $f_n = cy_n$. Let $z = c\Delta t$. The corresponding defect equation has solutions of the form $\delta_n = \beta^n$, and is therefore stable for $|\beta| \leq 1$. Which of the following equations relates β to z?

A: Unanswered
B: $0 = \beta^2 - (\alpha - 1)\beta - z$
C: $0 = \beta - (z^2 + z\alpha + \alpha)$
D: $0 = \beta - \frac{1+z\alpha/2}{1-z\alpha/2}$
E: $0 = \beta^2 - \beta(1 + \alpha z) - z(1 - \alpha)$
F: $0 = \beta^2(1 - \alpha^2) + 2\alpha z - 1$
G: $0 = \beta^2 - \alpha^2 z^2$
H: $0 = \beta^2(1 + \alpha z) - \alpha - z$

Question 27 By neglecting terms non-linear in Δy, rewrite the scalar implicit scheme

$$y_{n+1} = y_n + \Delta t \left[\frac{3}{4} f(y_n) + \frac{1}{4} f(y_{n+1})\right],$$

in the form

$$a\Delta y_n = \Delta t\, f(y_n),$$

where $\Delta y_n = y_{n+1} - y_n$. What is a?

A: Unanswered
B: 1
C: $(1 + \Delta t\, f_y)$
D: $(1 - \Delta t\, f_y)$
E: $(1 + \frac{3}{4}\Delta t\, f_y)$
F: $(1 - \frac{3}{4}\Delta t\, f_y)$
G: $(1 + \frac{1}{4}\Delta t\, f_y)$
H: $(1 - \frac{1}{4}\Delta t\, f_y)$

Question 28 Consider a (Runge-Kutta) numerical approximation to $y' = f(y)$:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}\Delta t\, [k_1 + k_2], \\ k_1 &= f(y_n), \\ k_2 &= f(y_n + 2\Delta t\, k_1). \end{aligned}$$

Which of the following is the $O(\Delta t^2)$ local truncation error of this scheme? [Notation: $f_y = \frac{df}{dy}(y_n)$.] [Hint: Use Taylor expansions. Compare to the analysis of predictor-corrector from the lectures.]

A: Unanswered
B: $\frac{1}{2}\Delta t^2 f_y f(y_n)$
C: $\frac{3}{2}\Delta t^2 f_y f(y_n)$
D: $\frac{5}{2}\Delta t^2 f_y f(y_n)$
E: $\frac{1}{2}\Delta t^2 f_y f(y_n)^2$
F: $\frac{3}{2}\Delta t^2 f_y f(y_n)^2$
G: $\frac{5}{2}\Delta t^2 f_y f(y_n)^2$