Applied Numerical Analysis – Homework # 4

Numerical Integration and Differentiation

Numerical Differentiation

Question 1 Identify the following difference schemes:

1. $$f'(x) \approx \frac{1}{h}\left[f(x+h) - f(x-h)\right].$$
2. $$f'(x) \approx \frac{1}{h}\left[f(x) - f(x-h)\right].$$
3. $$f'(x) \approx \frac{1}{h}\left[f(x+h) - f(x)\right].$$

as Forward Differences (FD), Backward Differences (BD), Central Differences (CD), or None of these.

A: 1. CD; 2. BD; 3. FD
B: 1. CD; 2. FD; 3. BD
C: 1. None; 2. BD; 3. FD
D: 1. None; 2. FD; 3. BD

Question 2 Given the following numerical differentiation schemes:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} + O(h)$$
$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} + O(h)$$
$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + O(h^2)$$

Use one of these formulas to determine, as accurately as possible, an approximation for $f'(0.0)$ using the data in the following table:

x	$f(x)$
0.0	1.0
0.3	1.3
0.6	1.8
0.9	2.4
1.2	3.3

The most accurate value for $f'(0.0)$ is:

A: 0.90 B: 1.00 C: 1.17 D: 1.35

Question 3 Given the numerical differentiation schemes and the table with data in Question 2. Use one of these formulas to determine, as accurately as possible, an approximation for $f'(1.2)$:

A: 2.46 B: 2.55 C: 2.87 D: 3.00

Question 4 Given the numerical differentiation schemes and the table with data in Question 2. Use one of these formulas to determine, as accurately as possible, an approximation for $f'(0.3)$:

A: 1.33 B: 1.36 C: 1.55 D: 1.82

Question 5 Given the following numerical differentiation schemes:

$$
\begin{aligned}
f'(x_0) &= \frac{f(x_0+h)-f(x_0)}{h} + O(h) \\
f'(x_0) &= \frac{f(x_0)-f(x_0-h)}{h} + O(h) \\
f'(x_0) &= \frac{f(x_0+h)-f(x_0-h)}{2h} + O(h^2) \\
f'(x_0) &= \frac{1}{2h}\Big[-3\,f(x_0)+4f(x_0+h)-f(x_0+2h)\Big] + O(h^2) \\
f'(x_0) &= \frac{1}{2h}\Big[3f(x_0)-4f(x_0-h)+f(x_0-2h)\Big] + O(h^2) \\
f'(x_0) &= \frac{1}{12h}\Big[f(x_0-2h)-8f(x_0-h)+8f(x_0+h)-f(x_0+2h)\Big] + O(h^4).
\end{aligned}
$$

Use one of these formulas to determine, as accurately as possible, an approximation for $f'(2.1)$ using the data in the following table:

x	$f(x)$
2.1	1.7
2.2	1.3
2.3	1.1
2.4	0.9
2.5	0.7
2.6	0.6

The most accurate approximation to $f'(2.1)$ is:

A: -4.5 B: -5.0 C: -5.5 D: -6.0

Question 6 Given the numerical differentiation schemes and the table with data in Question 5. Use one of these formulas to determine, as accurately as possible, an approximation for $f'(2.3)$:

A: -1.83 B: -1.44 C: -1.43 D: -1.33

Question 7 Given the numerical differentiation schemes and the table with data in Question 5. Use one of these formulas to determine, as accurately as possible, an approximation for $f'(2.6)$:

A: 0.0 B: -0.5 C: -0.6 D: -1.0

Question 8 Consider the one-sided difference formula for the 1st-derivative of $f(x)$:

$$f'(x) \approx \frac{1}{2h}\left[2f(x+3h)-7f(x+2h)+10f(x+h)-5f(x)\right].$$

What is the order of the truncation error of this approximation?

A: $\mathcal{O}(h)$

B: $\mathcal{O}(h^2)$
C: $\mathcal{O}(h^3)$
D: $\mathcal{O}(h^4)$
E: The approximation is exact.
F: The approximation is not consistent (error $\mathcal{O}(1)$).

Question 9 Consider approximating the first derivative of the function $f(x) = x^2$ at $x = 0$. Use two different approximations: (a) forward finite differences, and (b) central finite differences, with a stepsize of $h = 0.5$ in both cases. What is the approximation of the derivative in cases (a) and (b) respectively?

A: 0.25 and 0.00
B: 0.00 and 0.50
C: 0.00 and 0.25
D: 0.50 and 0.00
E: 0.00 and 0.00
F: 1.00 and 0.00
G: 0.50 and 0.50
H: None of these.

Question 10 The central difference formula for the first-derivative is

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f^{(3)}(\xi), \quad x - h < \xi < x + h.$$

Suppose that when evaluating $f(x+h)$ and $f(x-h)$ we encounter round-off errors, which are bounded by some number $\varepsilon = 5 \cdot 10^{-6}$; moreover, suppose that $f^{(3)}$ is bounded in a neighborhood of $x = 0.9$ by a number $M = 0.7$. We want to approximate $f'(0.9)$. Determine the optimal choice for the stepsize h.

A: 0.100
B: 0.025
C: 0.019
D: 0.001
E: 0.005
F: 0.050
G: 0.080
H: 0.028

Numerical integration

Question 11 Use an open Newton-Cotes formula with 2 nodes to approximate the integral

$$\int_0^1 \sqrt{1+x}\, dx$$

Hint: the nodes of an 2-point open Newton-Cotes formula are identical to the interior nodes of a 4-point closed Newton-Cotes formula.

A: 1.2228
B: 1.2190
C: 1.2071
D: 1.2189
E: 1.2247
F: 1.2203
G: 1.2059
H: 1.2208

Question 12 Find the degree of precision of the quadrature formula

$$\int_{-1}^1 f(x)\, dx \approx \frac{2}{27}\left[8f\left(-\tfrac{3}{4}\right) + 11f(0) + 8f\left(\tfrac{3}{4}\right)\right].$$

A: 0
B: 1
C: 2
D: 3
E: 4
F: 5
G: 6
H: 7

Question 13 Find the weights w_0 and w_1 such that the quadrature rule

$$\int_{-1}^1 f(x)\,\mathrm{d}x \approx w_0 f(-\tfrac{1}{2}) + w_1 f(\tfrac{1}{2})$$

has degree of precision 1.

A: $w_0 = w_1 = 2$
B: $w_0 = w_1 = 1$
C: $w_0 = w_1 = \frac{1}{2}$
D: $w_0 = w_1 = \frac{1}{4}$
E: $w_0 = -w_1 = 2$
F: $w_0 = -w_1 = 1$
G: $w_0 = -w_1 = \frac{1}{2}$
H: $w_0 = -w_1 = \frac{1}{4}$

Question 14 Find the weights w_0 and w_1 such that the quadrature rule

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx w_0 f(-a) + w_1 f(a)$$

has degree of precision at least 1 for any node locations $-a$, a.

A: $w_0 = w_1 = 2$
B: $w_0 = w_1 = 1$
C: $w_0 = w_1 = \frac{1}{2}$
D: $w_0 = w_1 = \frac{1}{a}$
E: $w_0 = -w_1 = 2$
F: $w_0 = -w_1 = 1$
G: $w_0 = -w_1 = \frac{1}{2}$
H: $w_0 = -w_1 = \frac{1}{a}$

Question 15 Find the value of a such that the quadrature rule

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx f(-a) + f(a)$$

has degree of precision 3.

A: $a = 1/\sqrt{2}$
B: $a = 1/\sqrt{3}$
C: $a = 1/2$
D: $a = 1$

Question 16 Use the trapezoidal rule to approximate the integral

$$\int_{\circ} 1 \cdot \, dx\, dy$$

where $\circ$ is the circle with origin $(x = 0, y = 0)$ and radius 1. [Hint: First convert the integral to polar coordinates r and θ, and then apply the trapezoidal rule to the r and θ integrals individually.]

A: 3.1227
B: 3.1333
C: 3.1416
D: 3.1523
E: 2.0503
F: 0.6738
G: 0.3333
H: 1.0472

Question 17 Consider the integral:

$$I = \int_{-1}^{1} x^2 + 1\, dx$$

Compute the exact value I of the integral, as well as the Trapezoidal Rule approximation I_3 with nodes $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$.

A: $I = 2.67$ and $I_3 = 2.85$
B: $I = 1.33$ and $I_3 = 1.33$
C: $I = 2.67$ and $I_3 = 3.00$
D: $I = 2.67$ and $I_3 = 2.67$
E: $I = 1.33$ and $I_3 = 1.35$
F: $I = 1.33$ and $I_3 = 1.00$
G: $I = 2.67$ and $I_3 = 2.50$
H: $I = 1.33$ and $I_3 = 1.52$

Question 18 Consider the integral:

$$I = \int_{-1}^{1} e^x \, dx$$

Compute the exact value I of the integral, as well as the Gaussian Quadrature approximation I_3. The Gaussian Quadrature nodes and weights given in the table.

i	x_i	w_i
1	-0.774597	0.555556
2	0.000000	0.888889
3	0.774597	0.555556

A: $I = 2.12$ and $I_3 = 2.12$
B: $I = 2.35$ and $I_3 = 2.37$
C: $I = 2.12$ and $I_3 = 2.15$
D: $I = 2.12$ and $I_3 = 2.14$
E: $I = 2.35$ and $I_3 = 2.38$
F: $I = 2.12$ and $I_3 = 2.11$
G: $I = 2.35$ and $I_3 = 2.35$
H: $I = 2.35$ and $I_3 = 2.33$

Question 19 Consider the integral:

$$I = \int_{0}^{1} e^x \, dx$$

Compute the exact value I of the integral, as well as the Gaussian Quadrature approximation I_3. The Gaussian Quadrature nodes and weights given in the table.

i	x_i	w_i
1	-0.774597	0.555556
2	0.000000	0.888889
3	0.774597	0.555556

A: $I = 2.35$ and $I_3 = 2.35$
B: $I = 2.35$ and $I_3 = 2.30$
C: $I = 1.72$ and $I_3 = 1.71$
D: $I = 2.35$ and $I_3 = 2.00$
E: $I = 1.72$ and $I_3 = 1.72$
F: $I = 1.72$ and $I_3 = 1.75$
G: $I = 2.35$ and $I_3 = 2.34$
H: $I = 1.72$ and $I_3 = 1.73$

Quiz 2013 – 45 mins

Question 20 **(Numerical diff.)** Consider the functions $f(x) = \sin(x)$ and $g(x) = x^2$. We use the forward difference scheme

$$f'(x) \approx D_F(f, x) = \frac{f(x+h) - f(x)}{h}$$

and the backward difference scheme

$$f'(x) \approx D_B(f, x) = \frac{f(x) - f(x-h)}{h},$$

to compute derivatives, with the same value of h in both cases. Which one of the following is true?

A: Unanswered
B: $D_F(f, 0) = D_B(f, 0)$ and $D_F(g, 0) = D_B(g, 0)$
C: $D_F(f, 0) = D_B(f, 0)$ and $D_F(g, 0) = -D_B(g, 0)$
D: $D_F(f, 0) = -D_B(f, 0)$ and $D_F(g, 0) = D_B(g, 0)$
E: $D_F(f, 0) = -D_B(f, 0)$ and $D_F(g, 0) = -D_B(g, 0)$
F: None of these.

Question 21 **(Numerical diff.)** Consider the function $f(x) = e^x$ which is differentiated using the forward difference scheme $D_F(f, x)$ (see e.g. Question 20). We consider all possible values of x and stepsize $h > 0$. Which one of the following is true of the error $\epsilon := f'(x) - D_F(f, x)$? [Hint: Examine the difference scheme applied to the function graphically.]

A: Unanswered
B: $\epsilon = 0$ always
C: $\epsilon > 0$ always
D: $\epsilon < 0$ always
E: ϵ can be positive or negative depending on h
F: ϵ can be positive or negative depending on x
G: ϵ can be positive or negative depending on both x and h

Question 22 **(Numerical diff.)** Apply the central difference scheme:

$$D_C(f, x) = \frac{f(x+h) - f(x-h)}{2h}$$

to a 2nd-degree polynomial (a parabola). Which one of the following is true of the error $\epsilon := f'(x) - D_C(f, x)$?

A: Unanswered
B: $\epsilon = 0$ always
C: $\epsilon > 0$ always
D: $\epsilon < 0$ always
E: ϵ can be positive or negative depending on h
F: ϵ can be positive or negative depending on x
G: ϵ can be positive or negative depending on both x and h

Question 23 **(Numerical diff.)** Consider this proposal for a finite difference scheme:

$$f'(x) = \frac{2f(x+2h) - f(x+h) - f(x)}{6h} + \epsilon.$$

How does the error ϵ behave in terms of h?

A: Unanswered
B: $\mathcal{O}(1)$ - inconsistent rule
C: $\mathcal{O}(h)$
D: $\mathcal{O}(h^{1.5})$
E: $\mathcal{O}(h^2)$
F: $\mathcal{O}(h^3)$

Question 24 **(Numerical int.)** Consider the quadrature rule:

$$\int_{-1}^{1} f(x)\,dx \approx Q(f, [-1, 1]) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right),$$

defined on the interval $[-1, 1]$. What is $Q(f, [0, 2])$, the approximation of the integral of:

$$f(x) = x^3$$

on the interval $[0, 2]$?

A: Unanswered
B: 0
C: 3
D: 4
E: $\frac{7}{2} + \frac{1}{\sqrt{3}}$
F: $\frac{9}{2} - \frac{1}{\sqrt{3}}$
G: $\frac{6}{\sqrt{3}}$
H: $\frac{7}{\sqrt{3}}$

Question 25 **(Numerical int.)** Consider the integration rule on the interval $[0, 2]$:

$$Q(f) = \frac{1}{3}f(0) + \frac{4}{3}f(1) + \frac{1}{3}f(2)$$

What is the degree of precision of this rule?

A: Unanswered
B: 0
C: 1
D: 2
E: 3
F: 4

Question 26 **(Numerical int.)** Find the node x_1, and the weight w_1 such that the quadrature formula

$$\int_0^1 f(x)dx \approx Q(f, [0, 1]) = \frac{1}{2} f\left(\frac{1}{3}\right) + w_1 f(x_1)$$

has the highest possible degree of precision. What is the value of x_1?

A: Unanswered
B: $\frac{2}{9}$
C: $\frac{1}{3}$
D: $\frac{4}{9}$
E: $\frac{1}{2}$
F: $\frac{5}{9}$
G: $\frac{2}{3}$
H: $\frac{7}{9}$

Question 27 **(Interpolation)** Function $f(x) = ax^3 + bx^2 + cx + d$ is a cubic polynomial ($a \neq 0$, $b \neq 0$). Six distinct locations $0 = x_0 < x_1 < \cdots < x_5 = 1$ are sampled from $f(x)$ and used to construct the following interpolants on the interval $[0, 1]$:

- a polynomial using the Lagrange basis
- a *natural* cubic spline (2nd-derivatives at the end points are zero)

Which of the following statements is true? On the interval $[0, 1]$:

A: Unanswered
B: both interpolants are identical to $f(x)$
C: only the Lagrange polynomial is identical to $f(x)$
D: only the natural cubic spline is identical to $f(x)$
E: neither interpolant is identical to $f(x)$