Applied Numerical Analysis – Homework #3

Interpolation: Splines, Multiple dimensions, Radial Bases, Least-Squares

Splines

Question 1 Consider a cubic spline interpolation of a set of data points, and derivatives of this spline at interior data points. The spline itself is continuous. Which of the following statements is true in addition?

A: None of the derivatives are continuous.
B: Only the 1st derivatives are continuous.
C: Only the 2nd derivatives are continuous.
D: Only the 1st and 2nd derivatives are continuous.
E: Only the 2nd and 3rd derivatives are continuous.
F: Only the 1st, 2nd and 3rd derivatives are continuous.
G: Cubic splines are infinitely differentiable.

Question 2 The following n data points, $(x_1, y_1),\ (x_2, y_2), \ldots,\ (x_n, y_n)$ are given. For conducting quadratic spline interpolation the x-data needs to be

A: equally spaced
B: in ascending or descending order
C: integers
D: positive

Question 3 The following incomplete y vs. x data is given:

x	1	2	4	6	7
y	5	11	??	??	32

The data is fit by quadratic spline interpolants given by

$$\begin{aligned} f(x) &= ax - 1, & 1 \le x \le 2 \\ f(x) &= -2x^2 + 14x - 9, & 2 \le x \le 4 \\ f(x) &= bx^2 + cx + d, & 4 \le x \le 6 \\ f(x) &= 25x^2 - 303x + 928, & 6 \le x \le 7 \end{aligned}$$

where a, b, c and d are constants. What is the value of c?

A: -303.00 B: -144.50 C: 0.0000 D: 14.000

Question 4 Which of the following functions is not a spline?

A:

$$f(x) = \begin{cases} x + 1 & x \in [0, 1] \\ 2 & x \in [1, 2] \end{cases}$$

B:

$$f(x) = \begin{cases} x^2 + 1 & x \in [0, 1] \\ 2x & x \in [1, 2] \end{cases}$$

C:

$$f(x) = \begin{cases} x^3 - 2x^2 + 3x & x \in [0, 1] \\ x^2 + 1 & x \in [1, 2] \end{cases}$$

D:

$$f(x) = \begin{cases} 2x^3 - 4\frac{1}{2}x^2 + 5x - 1\frac{1}{2} & x \in [0,1] \\ 3x^3 - 7\frac{1}{2}x^2 + 6x - \frac{1}{2} & x \in [1,2] \end{cases}$$

Bivariate Interpolation

Question 5 (4 pts) Given 4 data points (x_i, f_i), which form a rectangle (see table). Interpolate at the point $(x = 5/3, y = -4/3)$ using a bilinear interpolator. Hint: the bilinear interpolator, which interpolates a function f at the points $P_i = (x'_i, y'_i)$ with $P_1 = (-1,-1)$, $P_2 = (1,-1)$, $P_3 = (1,1)$, and $P_4 = (-1,1)$ is:

$$\phi(x',y') = \frac{f_1}{4}(1-x')(1-y') + \frac{f_2}{4}(1+x')(1-y') + \frac{f_3}{4}(1+x')(1+y') + \frac{f_4}{4}(1-x')(1+y')$$

.

i	1	2	3	4
x_i	1	2	2	1
y_i	-3	-3	-1	-1
f_i	1.3	1.5	0.3	0.2

A: 0.46
B: -0.21
C: -0.32
D: 2.52
E: 1.68
F: 1.12
G: -0.67
H: 1.00

Question 6 Given 4 data points (x_i, f_i), which form a rectangle (see table). Interpolate at the point $(x = 2/3, y = 4/3)$ using a bilinear interpolator.

i	1	2	3	4
x_i	0	1	1	0
y_i	0	0	2	2
f_i	1.7	3.9	0.6	1.6

A: 3.88
B: -0.32
C: -0.44
D: 2.53
E: 1.68
F: 1.95
G: 0.49
H: 2.00

Question 7 Linearly interpolate the data given in the table at the point $(x,y) = (2/3, 1)$.

i	1	2	3
x_i	0	1	1
y_i	0	1	2
f_i	2	3	1

A: 0.34
B: -1.00
C: 1.14
D: 1.95
E: 0.95
F: 2.00
G: 1.05
H: 2.20

Question 8 Linearly interpolate the function $f(x,y) = \sqrt{x+y}$ at the point $(x,y) = (2/3, 1/2)$ using the function values $f(x_i, y_i)$ at the points given in the table.

i	1	2	3
x_i	0	1	1
y_i	0	0	1

A: 0.78 | C: 0.87 | E: 1.71 | G: 0.72
B: 1.14 | D: 1.41 | F: 1.37 | H: 2.42

A: 0.78		C: 0.87		E: 1.71		G: 0.72	
B: 1.14		D: 1.41		F: 1.37		H: 2.42	

Question 9 The lift coefficient C_l is given for different angles of attack α and flap angles δ. Linearly interpolate the data given in the table, to determine the lift coefficient at $(\alpha, \delta) = (4, 1)$.

i	1	2	3
α_i	0	8	8
δ_i	0	0	4
$C_{l,i}$	0.20	1.00	1.16

A: 0.20
B: 0.86
C: 0.64
D: 0.45
E: 0.92
F: 0.48
G: 0.88
H: 0.52

Radial Basis Functions

Question 10 Given 4 data points (see table). We want to interpolate the data using the radial function $\varphi(r) = r$. The interpolation condition leads to a linear system of dimension 4, which can be written as $\mathbf{A}\,\mathbf{a} = \mathbf{f}$, where $\mathbf{a}$ contains the unknown coefficients and $\mathbf{f}$ is the vector with given function values $\mathbf{f} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \end{pmatrix}^T$. Compute the element $A_{2,3}$ of the matrix $\mathbf{A}$.

i	1	2	3	4
x_i	0	1	2	2
y_i	-4	0	1	2
f_i	-3	2	6	9

A: 0.00
B: 4.12
C: 5.10
D: 1.41
E: 1.00
F: 6.32
G: 0.12
H: 0.50

Question 11 Given data in the table, we want to interpolate the data using the radial function $\varphi(r) = e^{-r}$. The interpolation condition leads to a linear system of dimension 3, which can be written as $\mathbf{A}\,\mathbf{a} = \mathbf{f}$, where $\mathbf{a}$ contains the unknown coefficients and $\mathbf{f}$ is the vector with given function values $\mathbf{f} = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}^T$. We can obtain $\mathbf{a}$ by solving $\mathbf{a} = \mathbf{A}^{-1}\mathbf{f}$. Compute the matrix $\mathbf{A}$.

i	1	2	3
x_i	0	1	1
y_i	0	0	1
f_i	-2	-1	-3

A:

$$\mathbf{A} = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix};$$

B:

$$\mathbf{A} = \begin{bmatrix} 1.000 & 0.368 & 0.243 \\ 0.368 & 1.000 & 0.368 \\ 0.243 & 0.368 & 1.000 \end{bmatrix};$$

C:

$$\mathbf{A} = \begin{bmatrix} 1.000 & -0.378 & -0.146 \\ -0.378 & 1.000 & -0.378 \\ -0.146 & -0.378 & 1.000 \end{bmatrix};$$

D:

$$\mathbf{A} = \begin{bmatrix} 0.000 & 1.000 & 1.412 \\ 1.000 & 0.000 & 1.000 \\ 1.412 & 1.000 & 0.000 \end{bmatrix};$$

Least-Squares Regression

Question 12 The data in the following table is to be fitted with a linear function:

$$y(x) = ax + b.$$

Set up the normal equations for least squares and determine a and b.

x	y
0.1	21
0.2	18
0.4	15
0.5	15
0.8	12

The coefficients a and b are:

A: $a = -10.99$ and $b = 15.00$
B: $a = -0.12$ and $b = 20.50$
C: $a = 21.05$ and $b = 13.45$
D: $a = -12.00$ and $b = 21.00$
E: $a = -12.50$ and $b = 12.90$
F: $a = 11.95$ and $b = 0.46$
G: $a = 2.00$ and $b = 0.75$
H: $a = -9.12$ and $b = 1.16$

Question 13 You want to calibrate a pitot tube. From theory, the airspeed v is given by the pressure difference Δp according to the equation:

$$v(q) = a_1 \sqrt{\Delta p} + a_2,$$

where v is in [m/s] and Δp is in [bar]. Estimate the coefficients a_1 and a_2 from the given data, using least-squares.

Δp [bar]	v [m/s]
0.01	34
0.02	68
0.03	83
0.04	92

The coefficient a_1 is:

A: $a_1 = 412$
B: $a_1 = 581$
C: $a_1 = -20$
D: $a_1 = 212$
E: $a_1 = 128$
F: $a_1 = 75$
G: $a_1 = 220$
H: $a_1 = -18$

Question 14 The data in the following table is to be fitted with the following function:

$$y(x) = a_1 \frac{1}{1 + e^x} + a_2 x.$$

Set up the normal equations for least squares and determine a_1 and a_2.

x	y
-7	2.6
-3	2.8
-1	2.2
2	0.4
9	0.4

The coefficients a_1 and a_2 are:

A: $a_1 = 2.53$ and $a_2 = 0.07$
B: $a_1 = 2.11$ and $a_2 = 1.31$
C: $a_1 = -4.15$ and $a_2 = 1.05$
D: $a_1 = 3.02$ and $a_2 = 0.05$
E: $a_1 = -2.02$ and $a_2 = 3.11$
F: $a_1 = 4.19$ and $a_2 = 0.00$
G: $a_1 = -2.02$ and $a_2 = 0.00$
H: $a_1 = 4.19$ and $a_2 = 3.11$

Question 15 (5 pts) The upward velocity of a rocket has been measured with a tracking system as a function of time (see table). From theory, it is known that the velocity within the first 20 seconds after launch ($t = 0$ seconds) can be well approximated by a function:

$$v(t) = a_1 \ln\left(\frac{1}{1 - 0.02\,t}\right) + a_2\,t, \quad 0 \leq t \leq 30,$$

where v is in [m/s] and t is in [s]. Estimate the coefficients a_1 and a_2 from the given data using least-squares. (Hint: set-up the normal equations to 4 decimal places to reduce round-off errors.)

time t [s]	velocity $v(t)$ [m/s]
0	0
10	126.0
15	261.0
20	420.0

The coefficient a_1 is

A: $a_1 = 2436$
B: $a_1 = -68$
C: $a_1 = 3002$
D: $a_1 = 2109$
E: $a_1 = 1450$
F: $a_1 = -1020$

The Bigger Picture

Question 16 Suppose we define an approximating function $\phi(x) = \sum_{i=1}^{I} a_i\, b_i(x)$, where $b_i(x)$ are given functions of x. To determine the coefficients $\{a_i : i = 1 \dots I\}$ from given data $\{f_i : i = 1 \dots N\}$ at N distinct points, we can either use interpolation or least-squares approximation. Interpolation and least-squares approximation

A: always provide the same coefficients;
B: provide the same coefficients if the interpolation problem is uniquely solvable;
C: provide the same coefficients if the data are not noisy;
D: provide the same coefficients if $N = I$;
E: provide the same coefficients if $N = I$ and, at the same time, the data are not noisy;
F: never provide the same coefficients.

Question 17 A robot needs to follow a path that passes through six points. To find the shortest path that is also smooth you would recommend which of the following?

A: Pass a fifth order polynomial through the data
B: Pass linear splines through the data
C: Pass quadratic splines through the data
D: Regress the data to a second order polynomial

Quiz 2013 – 45 mins

Question 18 (Splines) We wish to construct a very smooth spline interpolant $p(x)$ on the interval $[a, b]$. We divide the interval up into sub-intervals based on the grid of 8 points:

$$a = x_0 < x_1 < \cdots < x_7 = b,$$

and approximate $p(x)$ as a different polynomial on each of these sub-intervals - i.e. a typical spline.

We require that $p(x)$ is continuous, and has continuous derivatives up to and including $p^{(5)}(x) = \frac{d^5 p}{dx^5}$ at all points in the interval. Furthermore we require that $p(x)$ interpolates the data points $f_0, \ldots, f_7$ at the grid locations. We impose no further boundary conditions.

How many constraints do these requirements imply on $p(x)$?

A: Unanswered
B: 33
C: 36
D: 42
E: 44
F: 49
G: 51
H: 56

Question 19 **(Splines)** The function $f(x) = a+bx+cx^2+dx^3+ex^4$ is a fourth order polynomial. We wish to interpolate this function. Two data sets, extracted from this function, are available:
Set 1

x_i	-2	-1	0	1	2
f_i	11	1	1	5	31

Set 2

x_i	-2	-1.5	0.5	1.5	2
f_i	11	3.44	1.94	13.19	31

Four interpolants are constructed labeled (i) to (iv):

(i) a fourth order polynomial using data set 1, using a Lagrange basis

(ii) a fourth order polynomial using data set 2, using a Newton basis

(iii) a cubic spline using data set 1, with natural boundary conditions

(iv) a cubic spline using data set 2, with natural boundary conditions

Which of these four interpolants are identical? [**Note: no calculation is necessary**].

A: Unanswered
B: All same
C: (i)=(ii), (iii)=(iv)
D: (i)=(iii), (ii)=(iv)
E: (i)=(ii)
F: (i)=(iii)
G: (iii)=(iv)
H: All different

Question 20 **(Splines)** Consider the data set

i	0	1	2
x_i	0	1	2
f_i	0	1	0

The data is fit by a natural cubic spline (i.e. the second derivatives at the boundaries are zero). What is the value of the spline at $x = \frac{1}{2}$? [Hint: use symmetry arguments to simplify the calculation.]

A: Unanswered
B: $\frac{1}{2}$
C: $\frac{3}{4}$
D: $\frac{5}{8}$
E: $\frac{7}{8}$
F: $\frac{9}{16}$
G: $\frac{11}{16}$
H: $\frac{13}{16}$

Question 21 **(Tensor-product interpolation)** We wish to interpolate the function of 10 variables $f(x_1, x_2, \ldots, x_{10})$, and use a quadratic polynomial approximation in each variable x_i. We construct the tensor-product grid. How many grid points do we obtain?

A: Unanswered	C: 30	E: 2048	G: 59049
B: 20	D: 1024	F: 19683	H: 177147

Question 22 **(Radial basis)** Interpolate the data given in the table using the radial function $\varphi(r) = r^2$. The interpolant is given by

$$p(\mathbf{x}) = a_0\varphi(|\mathbf{x} - \mathbf{x}_0|) + a_1\varphi(|\mathbf{x} - \mathbf{x}_1|) + a_2\varphi(|\mathbf{x} - \mathbf{x}_2|),$$

where $\mathbf{x} = (x, y)$ etc. What is the coefficient a_0, of the radial basis function centered at $\mathbf{x}_0$?

i	0	1	2
x_i	0	1	1
y_i	0	1	2
f_i	0	1	0

A: Unanswered	C: $\frac{1}{8}$	E: $\frac{3}{8}$	G: $\frac{5}{8}$
B: 0	D: $\frac{1}{4}$	F: $\frac{1}{2}$	H: $\frac{3}{4}$

Question 23 **(Radial basis)** Radial basis function interpolation is performed with the function:

$$\varphi(r) = \begin{cases} \frac{1}{2}(\cos(10\pi r) + 1) & \text{for } r < \frac{1}{10} \\ 0 & \text{for } r \geq \frac{1}{10} \end{cases}$$

It is applied to the data set defined on the unit square:

i	1	2	3	4
x_i	-1	1	1	-1
y_i	-1	-1	1	-1
f_i	1	2	3	4

What is the value of the interpolant at $(x, y) = (0, 0)$? [Note: *it is not necessary to construct the interpolant!*]

A: Unanswered	C: 1	E: 2.5	G: 4
B: 0	D: 2	F: 3	

Question 24 **(Interpolation)** You've learned how to find the interpolant of a data-set by setting up a set of linear interpolation conditions, and inverting the resulting matrix to find the coefficients a_i. Which of the following interpolants does not allow this approach?

A: Unanswered	E: $p(x, y) = a_1 + \ln(1 + a_2 x)$;
B: $p(x, y) = a_1 x + a_2 xy$;	F: $p(x, y) = a_1 + a_2 x^y$;
C: $p(x, y) = a_1 x^2 + a_2 x/y$;	G: $p(x, y) = a_1 x^{2.3} + a_2 y^{1.2}$;
D: $p(x, y) = a_1 \ln(x) + a_2 \frac{1}{\lvert y\rvert + x}$;	H: $p(x, y) = a_1 x^x + a_2 x^4$;

Question 25 **(Polynomial Interpolation)** The manufacturing cost p of a product depends on two parameters α and β. Use linear interpolation on the data in the table to approximate p.

i	0	1	2
α_i	0	80	120
β_i	0	0	5
p	0	40	70

If the price of product $p = 65$, what is the implied relationship between α and β?

A: Unanswered
B: $\alpha = 4\beta$
C: $\alpha = -4\beta$
D: $\alpha = 130 - 4\beta$
E: $\alpha = 130 + 4\beta$
F: $\alpha = \beta/4$
G: $\alpha = -\beta/4$
H: $\alpha = \beta = 0$