Applied Numerical Analysis – Homework #2

Polynomial Interpolation

Question 1 The problem of univariate polynomial interpolation using a grid N of distinct data points

A: Has no solution
B: Has always a unique solution
C: Has a unique solution if the Chebyshev grid is used
D: Has a unique solution if the underlying function f is continuously differentiable
E: Has a unique solution if the Chebyshev grid is used and the underlying function is continuously differentiable

Question 2 Univariate polynomial interpolation converges

A: Never
B: Always
C: Only for equidistant grids X if the underlying function is sufficiently smooth
D: Only for non-equidistant grids X if the underlying function is sufficiently smooth
E: Always for a Chebyshev grid
F: Always for a Chebyshev grid if the underlying function is sufficiently smooth

Question 3 A set of n data points (x_i, f_i), $i = 0, \ldots, n-1$ is given. What is the minimum degree of polynomial which is guaranteed to be able to interpolate all these points?

A: $n+2$
B: $n+1$
C: n
D: $n-1$
E: $n-2$
F: ∞

Question 4 What is the minimum order of the polynomial that interpolates the following points?

i	0	1	2	3
x_i	-2	-1	0	1
f_i	4	1	0	1

A: 0
B: 1
C: 2
D: 3
E: 4
F: ∞

Question 5 Which of the following functions can not be used as a basis for interpolation?

A: polynomial
B: rational functions
C: trigonometric
D: all of the above can be used

Question 6 Three data points (x_i, f_i) are given in the table below. The Newton representation of the interpolation polynomial is $p(x) = d_0 + d_1\, x + d_2\, x(x-1)$. Determine the coefficient d_2.

i	0	1	2
x_i	0	1	2
f_i	4	3	1

A: $d_2 = 4$ | C: $d_2 = -1/3$ | E: $d_2 = -2$
B: $d_2 = 3$ | D: $d_2 = 1/3$ | F: $d_2 = -1/2$

Question 7 Given the function $f(x) = cos(\pi x)$ for $x \in [0, \frac{1}{2}]$. Let $p(x)$ be the polynomial, which interpolates $f(x)$ at $x_0 = 0$ and $x_1 = \frac{1}{2}$. Determine the upper bound of the error $R(f; x) = f(x) - p(x)$ at $x = \frac{1}{4}$. Hint: suppose $f \in C^{n+1}([a, b])$. For any grid X of $n + 1$ nodes with $a = x_0 < x_1 < \ldots < x_n = b$ the interpolation error is

$$R(f; x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j), \quad \min_i(x_i, x) < \xi < \max_i(x_i, x).$$

A: 0.31 | C: 4.93 | E: 0.22
B: 0.62 | D: 0.03 | F: 0.02

Question 8 When interpolating a smooth function with polynomials on an interval $[-1, 1]$, a grid $X = (x_0, x_1, \ldots, x_n)$ is choosen. Which of the following choices of grid x_i do you expect to give a result with minimum interpolation error for (i) a Lagrange basis, and (ii) a monomial basis?

1. scattered
2. equidistant
3. a higher concentration of points around the center of the domain.
4. a higher concentration of points near the edges of the domain.

A: (i) 1, (ii) 3. | C: (i) 2, (ii) 2. | E: (i) 3, (ii) 2. | G: (i) 4, (ii) 3.
B: (i) 2, (ii) 1. | D: (i) 2, (ii) 3. | F: (i) 4, (ii) 2. | H: (i) 4, (ii) 4.

Question 9 Suppose we want to interpolate a function $f(x)$ on the interval $[1, 5]$ using a cubic polynomial. The grid X is given by $1 < x_0 < x_1 < \ldots < x_n < 5$. We want to use a Chebyshev grid. Compute the node x_0. Hint: the zeros of the Cheybshev polynomial of degree m, $T_m(x)$, are given by

$$\xi_i = \cos\left(\frac{2i-1}{2m}\pi\right), \quad i = 1, \ldots, m.$$

A: -0.924 | C: 4.848 | E: 1.152 | G: 1.098
B: 2.618 | D: 2.235 | F: 1.000 | H: 1.268

Question 10 Consider a function $f(x)$ defined on the interval $[0, 1]$, which is approximated by Lagrange interpolant $p(x)$ constructed on a uniform grid with $n + 1$ points $x_i = i/n$. Which of the following statements is true in this situation?

1. Any number of data points n can be interpolated exactly
2. The interpolatng polynomial passes through the given data points.
3. As $n \to \infty$ we can be certain that the error $|p(x) - f(x)| \to 0$ for all $x \in \mathbb{R}$.
4. As $n \to \infty$ we can be certain that the error $|p(x) - f(x)| \to 0$ for all $x \in [0, 1]$.

A: 1, 2 | C: 1, 4 | E: 1, 2, 3 | G: 2, 3, 4
B: 2, 3 | D: 3, 4 | F: 1, 3, 4 | H: 1, 2, 3, 4

Question 11 A set of data points (x_i, f_i) is interpolated with polynomial interpolation. Which of the following has an influence on the interpolating polynomial?

1. The choice of basis (monomial, Lagrange, Newton, etc.)
2. The ordering of the points x_i.
3. The locations x_i.
4. The values f_i.

A: 1, 2
B: 2, 3
C: 1, 4
D: 3, 4
E: 1, 2, 3
F: 1, 3, 4
G: 2, 3, 4
H: 1, 2, 3, 4

Question 12 Consider the function $f(x) := \cos(\pi x)$ defined for $x \in [0, \frac{1}{2}]$. Let $p_1(x)$ be a first-order polynomial interpolating $f(x)$ at the nodes $x_0 = 0$ and $x_1 = \frac{1}{2}$. What is the exact error in the interpolant $\epsilon(x) = |f(x) - p_1(x)|$ at $x = \frac{1}{4}$?

A: 0.2071
B: 0.6169
C: 4.9348
D: 0.0312
E: 0.2181
F: 0.0221

Question 13 The following data of the velocity of a body is given as a function of time.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

Approximate the velocity at $16s$ by interpolating with a linear polynomial.

A: 22.33
B: 24.33
C: 25.33
D: 26.33
E: 27.33
F: 28.33

Question 14 The following data of the velocity of a body is given as a function of time.

Time (s)	0	1	3	4	5
Velocity (m/s)	10	11	14	25	77

Approximate the velocity at $2s$ by interpolating with a quadratic polynomial using the 1st three data points. [Tip: The calculation is easiest with a Newton basis.]

A: 12.1333
B: 12.2333
C: 12.3333
D: 12.4333
E: 12.5333
F: 12.6333

Question 15 Determine the Lagrange interpolation polynomial given the following data set:

i	0	1	2	3
x_i	-1	1	3	5
$f(x_i)$	-6	0	-2	-12

A: $p(x) = -x^2 + 3x - 2$
B: $p(x) = -4x^2 + 10x - 6$
C: $p(x) = \frac{1}{8}x^3 - \frac{9}{8}x^2 + \frac{23}{8}x - \frac{15}{8}$
D: $p(x) = \frac{5}{24}x^3 - \frac{39}{24}x^2 + \frac{67}{24}x - \frac{33}{24}$

Question 16 The following x, y data is given:
The second-order polynomial in a Newton basis for the above data is given by

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

The value of a_2 is

x	15	18	22
y	24	37	25

A: -1.0476 B: -4.3333 C: -3.0000 D: -0.1429

Quiz 2013 – 45 mins

Question 17 A function $f(x)$ is interpolated on the interval $4 \leq x_0 \leq x_1 \leq ... \leq x_n \leq 5$ using a polynomial of unknown degree. A Chebyshev grid was used and $x_0 = 4.0170$. What was the degree of the polynomial used for this interpolation? [Hint: The zeros of the Chebyshev polynomial of degree p on the interval $[-1, 1]$ are given by:

$$\xi_i = \cos\left(\frac{2i-1}{2p}\pi\right), \quad i = 1, 2, \ldots, p].$$

A: Unanswered
B: 2
C: 3
D: 4
E: 5
F: 7
G: 10
H: 11

Question 18 Consider the function $f(x) = x^3$. Let $p_2(x)$ be a second order polynomial which interpolates $f(x)$ at $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. We define the L_1-norm of the interpolation error on $[a, b]$ as:

$$\varepsilon = \int_a^b |p_2(x) - f(x)| dx.$$

What is the value of ε on the interval $[0, 1]$?

A: Unanswered
B: 0
C: $-\frac{1}{3}$
D: $\frac{1}{4}$
E: $-\frac{1}{3}$
F: $\frac{1}{3}$
G: $-\frac{8}{3}$
H: $\frac{8}{3}$

Question 19 By Cauchy's theorem the interpolation error for a degree n-polynomial is

$$R_n(f; x) := f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x),$$

for suitable $\xi(x)$ and ω. Based on this formula, for the interpolation problem of Question 18, what is the maximum interpolation error of any point in $[0, 1]$?

A: Unanswered
B: $\sqrt{3}/9$
C: $2\sqrt{3}/9$
D: $3\sqrt{3}/9$
E: $4\sqrt{3}/9$
F: $\sqrt{3}/3$
G: $2\sqrt{3}/3$
H: $3\sqrt{3}/3$

Question 20 Consider polynomial interpolant $p_n(x)$ of the function

$$f(x) = |x|$$

on the interval $x \in [-1, 1]$, using a Chebychev grid with $n + 1$ nodes. Which one of the following statements is true?

A: Unanswered
B: By the Weierstrass theorem there exists a polynomial such that the interpolation error is less than ϵ for any $\epsilon > 0$.
C: The interpolant $p_n(x)$ disagrees with $f(x)$ everywhere because $f(x)$ is not differentiable.

D: By Cauchy interpolation error theorem the interpolation error is

$$R_n(f;x) := f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega_{n+1}(x),$$

where $\omega_{n+1}(x)$ is the nodal polynomial.

E: The maximum interpolation error in the interval grows as n increases.

Question 21 A ball is dropped from rest at an altitude of 300 meters in air. Approximate its height as a function of time $f(t)$ by a Taylor expansion about time $t = 0$ truncated after the quadratic term. At what time does it hit the ground? [Assume a gravitational acceleration of $10\,ms^{-2}$.]

A: Unanswered
B: $5.75\,s$
C: $5.82\,s$
D: $5.98\,s$
E: $7.48\,s$
F: $7.75\,s$
G: $7.82\,s$
H: $7.98\,s$

Question 22 Consider Question 21 again. We would like to correct for air resistance. At $t = 8\,s$ an exact measurement of the velocity is made. The ball is dropping at $f'(8) = -60.8\,ms^{-1}$. Approximate the path of the ball by the cubic

$$p(x) = at^3 + bt^2 + ct + d.$$

What is the value of a, i.e. the cubic correction for air resistance? [Hint: Set up 4 interpolation conditions.]

A: Unanswered
B: 0
C: 0.01
D: 0.02
E: 0.05
F: 0.1
G: 0.2
H: 0.5

Question 23 Consider once more the falling ball from Question 21. An even better model correcting for air-resistance is

$$f(t) = a - b\ln\left(\cosh\left(\frac{t}{8}\right)\right)$$

where a and b are constants to be determined. We know the initial height is *precisely* $300\,m$. Furthermore we have two *approximate* measurements of position at later times:

t	height
4	220
8	22

What is the best choice for b in a least-squares sense? [Note: The hyperbolic cosine is defined $\cosh x = \frac{e^x + e^{-x}}{2}$.]

A: Unanswered
B: 583
C: 593
D: 603
E: 613
F: 639
G: 641
H: 643

Question 24 Perform least-squares regression with a *linear* regressor for the data set:

x	height
0	0.1
1	0.2
2	-0.1
4	0.1

At what location does the regressor have a root?

A: Unanswered	C: 2	E: 4	G: 6
B: 1	D: 3	F: 5	H: 7