Applied Numerical Analysis – Example Problems #1

Preliminaries, Taylor Expansions, Errors, Solving Non-linear Equations

Floating Point

Question 1 Consider the positive floating point number system $s \times b^e$, where $b = 10$, s is a 5-digit significant $1.0000 \leq s \leq 9.9999$ and $-8 \leq e \leq 8$. The system is completed with the number 0. What is the machine epsilon (i.e. the smallest number which, added to 1, is distinct from 1)?

A: 1×10^{-4} B: 9.9999×10^{-5} C: 1×10^{-5} D: 1×10^{-8}

Question 2 What is the total number of *distinct* numbers in the number system described in Question 1, including zero?

A: 99999 B: 1530001 C: 1710001 D: 999990001

Question 3 What is the result of the following algorithm, where x is computed using the number system described in Question 1:

$x \leftarrow x_0$
for $i = 1$ **to** 100 **do**
  $x \leftarrow \sqrt{x}$
end for
for $i = 1$ **to** 100 **do**
  $x \leftarrow x^2$
end for

for the case when $x_0 > 1$ and $0 < x_0 < 1$, respectively? (Rounding: assume that $\sqrt{x}$ and x^2 are performed in exact arithmetic and then rounded to the nearest representable number.)

A: 0, 0 C: 1, 0 E: 0, Overflow G: 0, x
B: 0, 1 D: 1, 1 F: 1, Overflow H: x, x

Taylor expansion, truncation error

Question 4 What is the third non-zero term in the Taylor expansion of $\cos(x)$ about $x = 0$?

A: $\frac{1}{4!}x$ C: $\frac{1}{4!}\cos(x)$ E: $-\frac{1}{2!}x^2$
B: $\frac{1}{4!}x^4$ D: $\frac{1}{4!}\cos(x)x^4$

Question 5 Represent $f(x) = e^x$ by a 3-term truncated Taylor expansion about $x = 1$. What is the first term in the truncation error?

A: $\frac{(x-1)^3}{3}e$ C: $\frac{x-1}{6}e$ E: $\frac{(x-1)^2}{2}$ G: $\frac{x-1}{3}$
B: $\frac{x-1}{3}e$ D: $\frac{(x-1)^3}{6}e$ F: $\frac{(x-1)^3}{3}$ H: $\frac{x-1}{6}$

Question 6 The function $f(x) = \exp(2x)$ is written as a 3-term Taylor expansion $P(x)$ about $x = x_0$, plus an exact remainder term $R(x)$, so that:

$$f(x) = P(x) + R(x).$$

What is the Lagrange form of the remainder $R(x)$? (Where in the following $\xi \in [x_0, x]$, and $h = x - x_0$.)

A: $\frac{1}{3}\exp(2\xi)(x_0+h)^3$
B: $\frac{4}{3}\exp(2\xi)h^3$
C: $\frac{1}{6}\exp(2h)\xi^3$
D: $\frac{8}{6}(h\xi)^3$

Question 7 Write $\sin(x)$ as a truncated Taylor series expansion about $x = 0$ with two *non-zero* terms. What is the magnitude of the first non-zero term in the truncation error at $x = \frac{\pi}{2}$?

A: 0.07969
B: 0.02
C: 0.008727
D: 0.008333

Question 8 Approximate e^{-x^2} by a 2-term Taylor series expansion about $x = 1$. What is the magnitude of the first term in the truncation error at $x = 0$?

A: e^{-1}
B: $2e^{-1}$
C: $\frac{1}{2}e^{-1}$
D: $\frac{1}{3}e^{-1}$

Root-finding

Question 9 What is the approximation of the root of the function $f(x) = e^x - 1$, if three steps of repeated bisection are applied on a starting interval $[x_1, x_2] = [-2, 1]$? (The root approximation is the center of the remaining interval.)

A: -0.5
B: -0.39346
C: -0.125
D: 0
E: 0.1331
F: 0.75

Question 10 Assume that a function $f(x)$ has multiple roots in the interval $[a, b]$, and $f(a) > 0$, $f(b) < 0$. Repeated bisection is applied, starting on this interval. How will the iteration behave? (Hint: perform the algorithm graphically on a suitable curve.) It will:

A: Take no steps.
B: Fail to converge.
C: Converge to one of the roots.
D: Converge to more than one root.
E: Terminate with a solution which is not a root.
F: Terminate with an interval containing all roots.

Question 11 Rearrange the function $f(x) = e^x - 5x^2$ (without adding terms) into a suitable format for fixed-point iteration. Make sure the iteration converges to a root, starting at an initial guess of $x_0 = 10$. What is the estimate of the root after two iterations of your method? (You may need to try more than one choice of fixed-point iteration.)

A: 0
B: 0.447
C: 1.156
D: 4.708
E: 5.263
F: 9.572

Question 12 Given a particular fixed-point iteration $x_{i+1} = g(x_i)$ which is known to converge, how do you expect the error ϵ of consecutive iterations to be related? (K is some constant with $|K| < 1$.)

A: $\epsilon_{i+1} < K\epsilon_i$
B: $\epsilon_{i+1} < K\epsilon_i^3$
C: $\epsilon_{i+1} > K\epsilon_i$
D: $\epsilon_{i+1} > K\epsilon_i^2$
E: $\epsilon_{i+1} < K + \epsilon_i$
F: $\epsilon_{i+1} > K + \epsilon_i^2$

Question 13 Apply Newton's method to $g(x) = 8^x - 8x^3$. Algebraically there is a root at $x = 1$. What is the absolute error to this root after a single iteration, using an initial guess of $x_1 = 0.5$?

A: -15.436 B: 0 C: 0.435 D: 0.565 E: 14.936

Question 14 Does the Newton iteration in Question 13 converge using the given initial guess?

A: Yes, the iterations converge to $x = 1$.
B: Yes, although it doesn't converge to $x = 1$.
C: No, Newton's method diverges using this guess.
D: No, the iterations oscillate forever.
E: More information is needed to answer this question.

Question 15 Consider the two-variable problem consisting of two scalar equations:

$$x^2 + y^2 = 1, \quad xy = \frac{1}{4}$$

Newton's method is applied to solve this system. Which of the following represents the first iteration of Newton's method for this system, with an initial guess of $x_0 = 2, y_0 = -1$? (Where $\Delta x_0 = x_1 - x_0$, and similarly for y.) [Hint: For multiple equations and variables the derivatives form a matrix.]

A: $\frac{4\Delta x_0 - 1\Delta y_0}{4} = 0, \ \frac{-2\Delta x_0 + 2\Delta y_0}{-2.25} = 0$
B: $\frac{4\Delta x_0 - 1\Delta y_0}{\Delta x_0} = -2.25, \ \frac{-2\Delta x_0 + 2\Delta y_0}{\Delta y_0} = 4$
C: $4\Delta x_0 - 1\Delta y_0 = 4, \ -2\Delta x_0 + 2\Delta y_0 = -2.25$
D: $4\Delta x_0 - 2\Delta y_0 = -4, \ -1\Delta x_0 + 2\Delta y_0 = 2.25$

Question 16 What is the behaviour of Newton's method for the equations given in Question 15, and an initial condition for which $x_0 = y_0$?

A: Converges to a root with a quadratic rate of convergence.
B: Converges to a root with only a linear rate of convergence.
C: Converges in 1 iteration.
D: Converges to a false root at $x = 0$, $y = 0$.
E: Diverges in 1 iteration.
F: Diverges in several iterations.

Quiz 2013 – 45 mins

Question 17 Find the first 3 terms of the Taylor expansion of $f(x) = e^x$ about $x_0 = 0$. These first 3 terms can be considered an approximation of $f(x)$ near 0. What is the value of this approximation at $x = 1$?

A: 3.0
B: 2.7182818
C: 2.5
D: 2.0
E: 1.5
F: 1.0

Question 18 Consider the positive floating point number system $s \times b^e$, where $b = 10$, s is a 16-digit significant $1 \leq s \leq 10 - 1 \times 10^{-15}$ and $-300 \leq e \leq 300$. This is close to the double-precision standard. What is the machine epsilon in this system (i.e. the smallest number which, when added to 1, gives a result distinct from 1)?

A: 1×10^{-7}
B: 1×10^{-8}
C: 1×10^{-13}
D: 1×10^{-14}
E: 1×10^{-15}
F: 1×10^{-16}
G: 1×10^{-299}
H: 1×10^{-300}

Question 19 Consider the floating-point system from Question 18. Recursive bisection is applied in exact arithmetic to find a root of a function $f(x)$ starting on the interval $[0, 1]$. How many iterations can be performed before the interval width is smaller than the smallest number respresented in the floating-point system?

A: 4	C: 49	E: 449	G: 4449
B: 9	D: 96	F: 996	H: 9996

Question 20 Consider the floating-point system from Question 18. Newton's method is now applied in exact arithmetic to find a root of a function $f(x)$. Let the initial error be $\epsilon_0 = 0.1$, and assume Newton's method converges quadratically from here. Approximately how many iterations are required before the interval width is smaller than the smallest number respresented in the floating-point system?

A: 4	C: 49	E: 449	G: 4449
B: 9	D: 96	F: 996	H: 9996

Question 21 Consider once more the floating point number system from Question 18, this time completed with the number 0, a sign bit, and an infinity condition `inf`. Newton's method is applied *in this system* to the function $f(x) = x^2 + \frac{1}{x}$, which has a unique root at $x = -1$, a zero derivative at $\hat{x} = \sqrt[3]{\frac{1}{2}}$ and a singularity at $x = 0$. Which one of the following statements is true?

A: The singularity causes divergence for any initial guess x_0.
B: The finite accuracy of the system means the iteration will always converge.
C: The iteration will not converge for $x_0 = 0$.
D: The iteration will not converge for $x_0 = \hat{x}$.
E: The iteration will not converge for both $x_0 = 0$ and $x_0 = \hat{x}$.
F: None of the above.

Question 22 Consider the function $f(x) = (x + 2)(x - 1)(x - 3)$. We are interested in finding the root closest to zero. We apply repeated bisection with the following starting intervals:

1. $[-1, 2]$
2. $[0, 8]$
3. $[-3, 0]$
4. $[-5, 5]$

For exactly which of these intervals does the method converge to $x = 1$? [It may be helpful to sketch the curve and apply the method by hand to see what happens.]

A: 1 only	C: 2,3,4 only	E: 1,4 only	G: All
B: 2,3 only	D: 4 only	F: 1,2,3 only	H: None

Question 23 A fixed point iteration is specified as:

$$x_{i+1} = x_i^2 - 2x_i + 1.$$

What equation is being solved here?

A: $0 = x_i^2 - x_i + 1$	D: $0 = x_i^2 + x_i + 1$
B: $0 = x_i^2 - 2x_i + 1$	E: $0 = x_i^2 + 1$
C: $0 = x_i^2 - 3x_i + 1$	F: $0 = x_i^3 - x_i^2 + x_i$

Question 24 Apply a single iteration of Newton's method to the function $f(x) = 2^x - 1$, using an initial guess of $x_0 = 1$. What is the updated estimate of the root, x_1?

A: 0.279	C: 0.297	E: 0.419
B: 0.289	D: 0.393	F: Diverges