# What to know

- Specifying type of ODE
- Convert high-order ODEs to system of first order ODEs
- Graphical interpretation of ODE integration schemes
- Implementation of given ODE integration schemes
- Determination order of convergence ODE integration schemes
- Determination stability regions ODE integration schemes

TUDelft

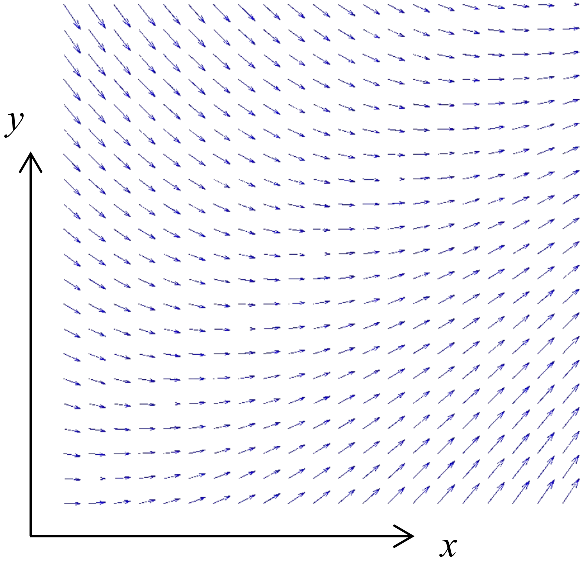
# Ordinary differential equations

- First order: highest derivative is first derivative y'(x)
- Ordinary: y = y(x), i.e. function of one variable x

$$y'(x) = f\left(x, y(x)\right), \quad x \in I = [x_0, x_n]$$

- In general, infinitely many solutions y(x)

$$y' = \frac{x - y}{2}$$



# Ordinary differential equations

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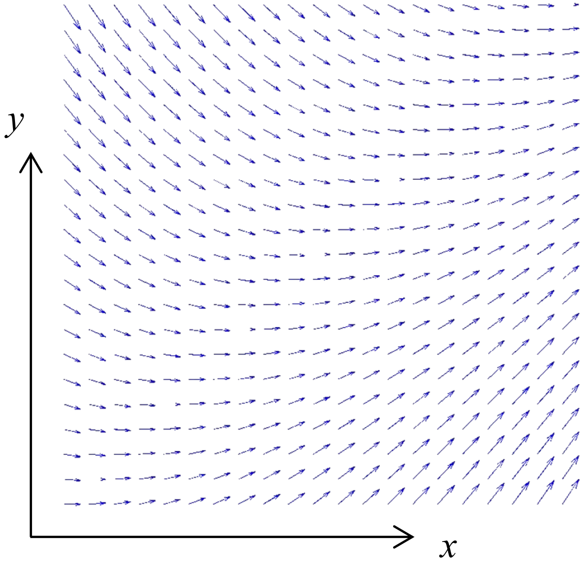
- In general, infinitely many solutions y(x)
- Initial / boundary conditions need to be specified for a single solution

initial condition $y\left(x_0\right)=y_0$

boundary condition $g\left(y\left(x_0\right),y\left(x_n\right)\right)=0$

$$y'=\frac{x-y}{2}$$



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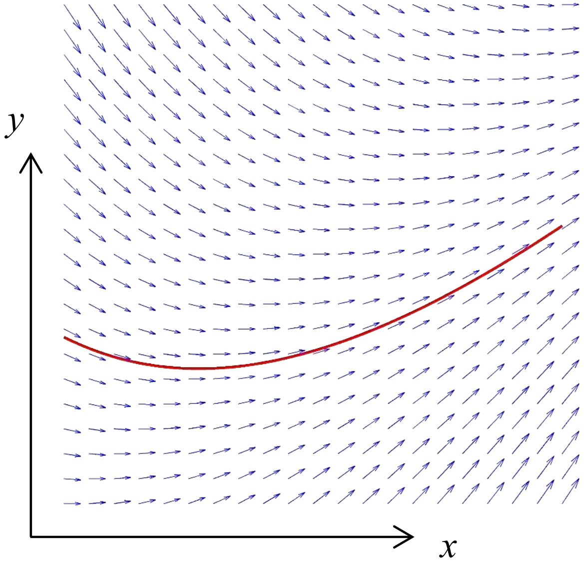
**initial value problem**

boundary value problem

$$y(x_0) = y_0$$

$$g\left(y(x_0), y(x_n)\right) = 0$$

$$y' = \frac{x-y}{2}$$



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# Ordinary differential equations

- System of p first-order ordinary differential equations, i.e.

$$\vec{y}'(x) = \vec{f}\left(x, \vec{y}(x)\right) \quad \left\{ \begin{array}{lcl} y_1'(x) & = & f_1\left(x, y_1(x), y_2(x), \ldots, y_p(x)\right) \\ y_2'(x) & = & f_2\left(x, y_1(x), y_2(x), \ldots, y_p(x)\right) \\ & \vdots & \\ y_p'(x) & = & f_p\left(x, y_1(x), y_2(x), \ldots, y_p(x)\right) \end{array} \right.$$

initial conditions $\vec{y}(x_0) = \vec{y}_0 = \begin{pmatrix} y_1(x_0) \\ \vdots \\ y_p(x_0) \end{pmatrix}$

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# Ordinary differential equations

- m-th order ordinary differential equation

$$y^{(m)}(x) = f\left(x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right)$$

initial conditions $y^{(i)}(x_0) = y_0^{(i)}, \quad i = 0,1,\ldots,m-1$

- Transform m-th order ordinary differential equation to an equivalent system of m first order ordinary differential equations by using auxiliary functions:

$$\left.\begin{array}{ccc} z_1(x) & = & y(x) \\ z_2(x) & = & y'(x) \\ & \vdots & \\ z_m(x) & = & y^{(m-1)}(x) \end{array}\right\} \qquad \begin{pmatrix} z_1' \\ \vdots \\ z_{m-1}' \\ z_m' \end{pmatrix} = \begin{pmatrix} z_2 \\ \vdots \\ z_m \\ f\left(x, z_1, z_2, \ldots, z_m\right) \end{pmatrix}$$

# Ordinary differential equations

- Newton's second law of motion: $my''(t) = mg - ky(t)$
- 2nd order ordinary differential equation
- Introduce the auxiliary functions:

$$\underbrace{\begin{aligned} z_1(t) &= y(t) \\ z_2(t) &= y'(t) \end{aligned}}$$

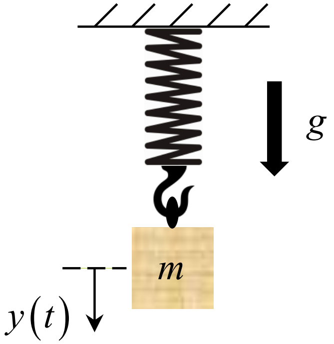
- System of 2 first-order ODEs:

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} z_2 \\ g - (k/m) z_1 \end{pmatrix}$$

- Initial conditions:

$z_1(0) = y(0)$ Initial position

$z_2(0) = y'(0)$ Initial velocity



$$\underbrace{\vec{z}'(t) = \begin{bmatrix} 0 & 1 \\ -(k/m) & 0 \end{bmatrix} \vec{z}(t) + \begin{pmatrix} 0 \\ g \end{pmatrix}}$$

$$\vec{z}'(t) = A\,\vec{z}(t) + \vec{b}$$

$$\vec{z}'(0) = \vec{z}_0$$

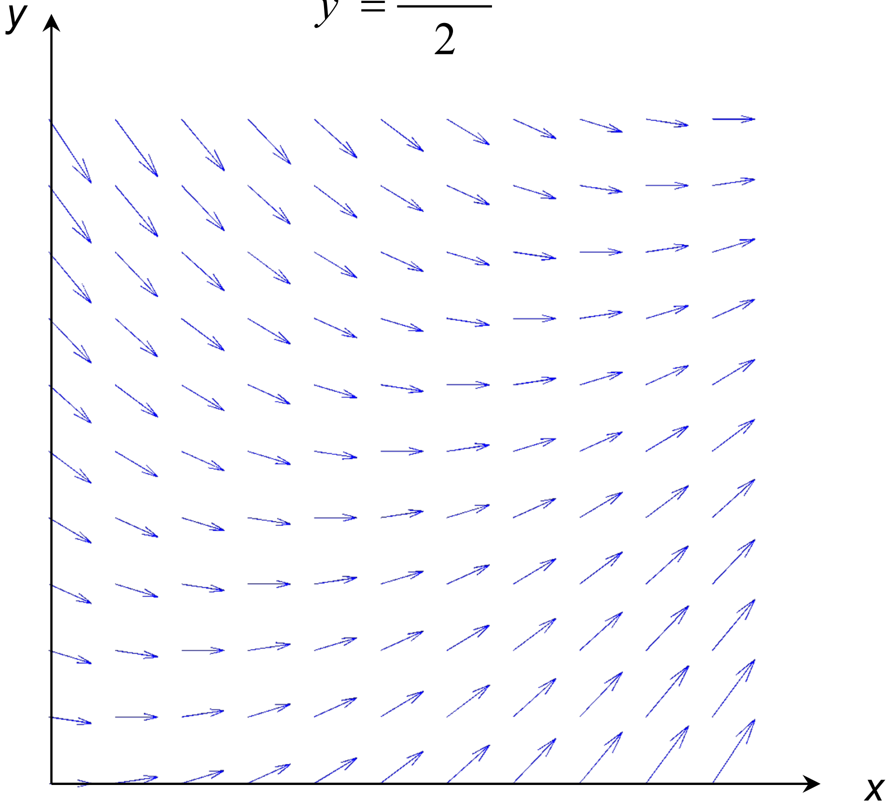
Linear system

# Solving ODEs

## Forward Euler-Cauchy

$$y(x) = C\exp[-x/2] + x - 2$$

$$y' = \frac{x-y}{2}$$



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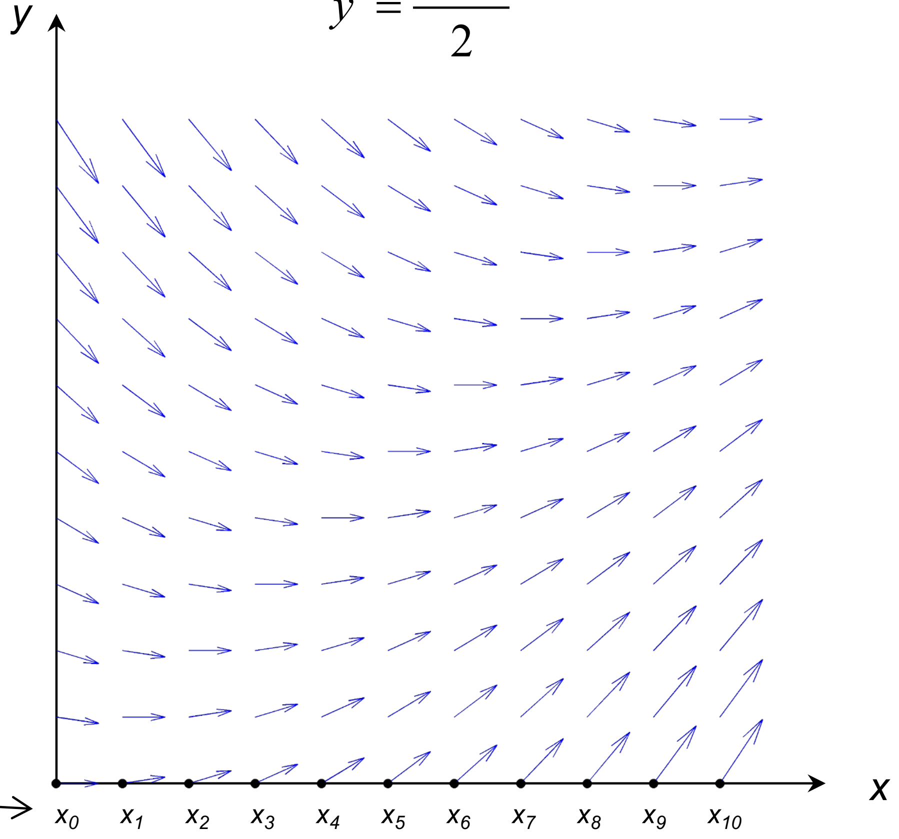
# Solving ODEs

## Forward Euler-Cauchy

$$y' = \frac{x-y}{2}$$



split up interval

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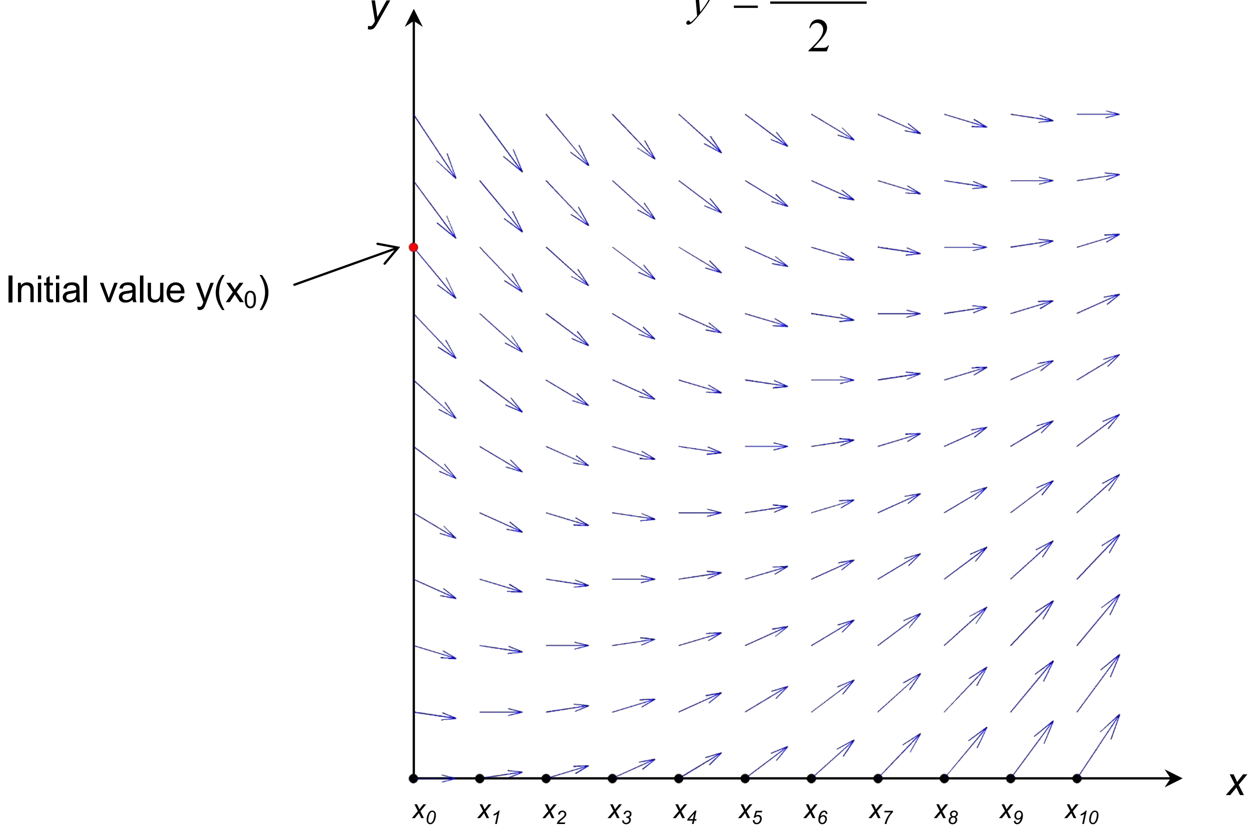
# Solving ODEs

## Forward Euler-Cauchy

$$y(x) = 4.4\exp[-x/2] + x - 2$$

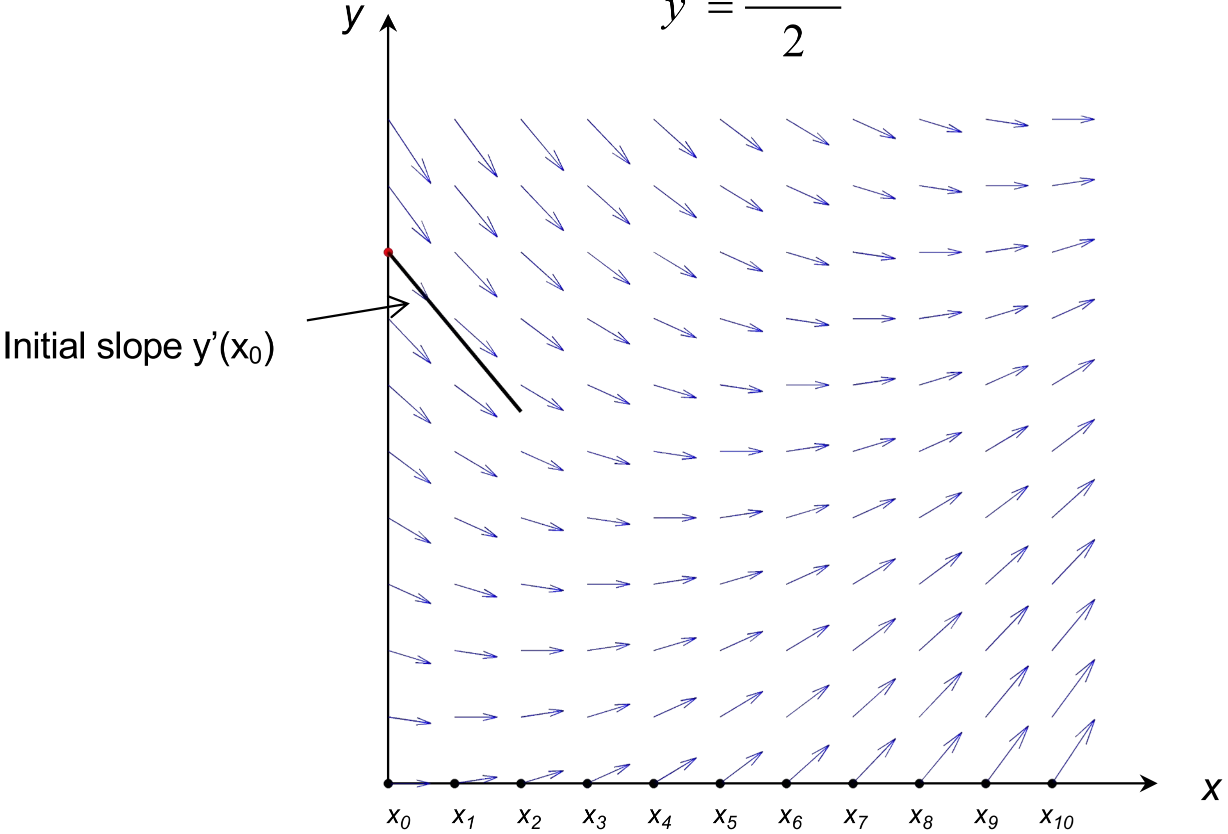
$$y' = \frac{x-y}{2}$$



# Solving ODEs

## Forward Euler-Cauchy

$$y' = \frac{x-y}{2}$$



# Solving ODEs

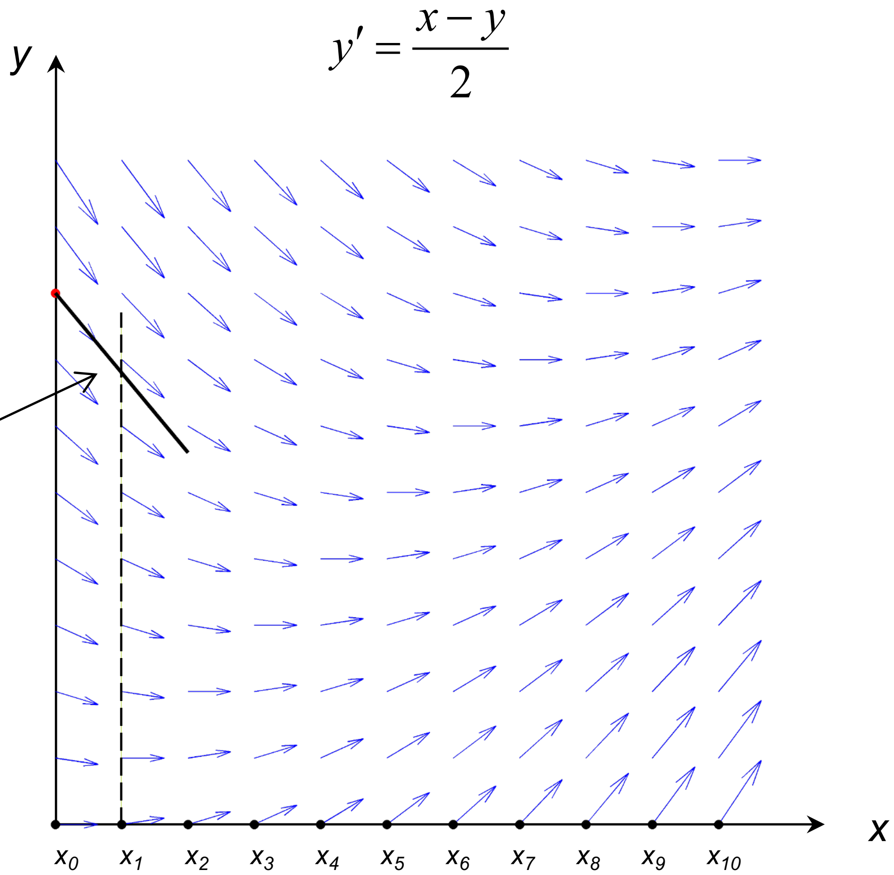
## Forward Euler-Cauchy

Approximate slope at $x_0$ with forward-difference:

$$y'(x_0) \approx \frac{y(x_0+h)-y(x_0)}{h}$$

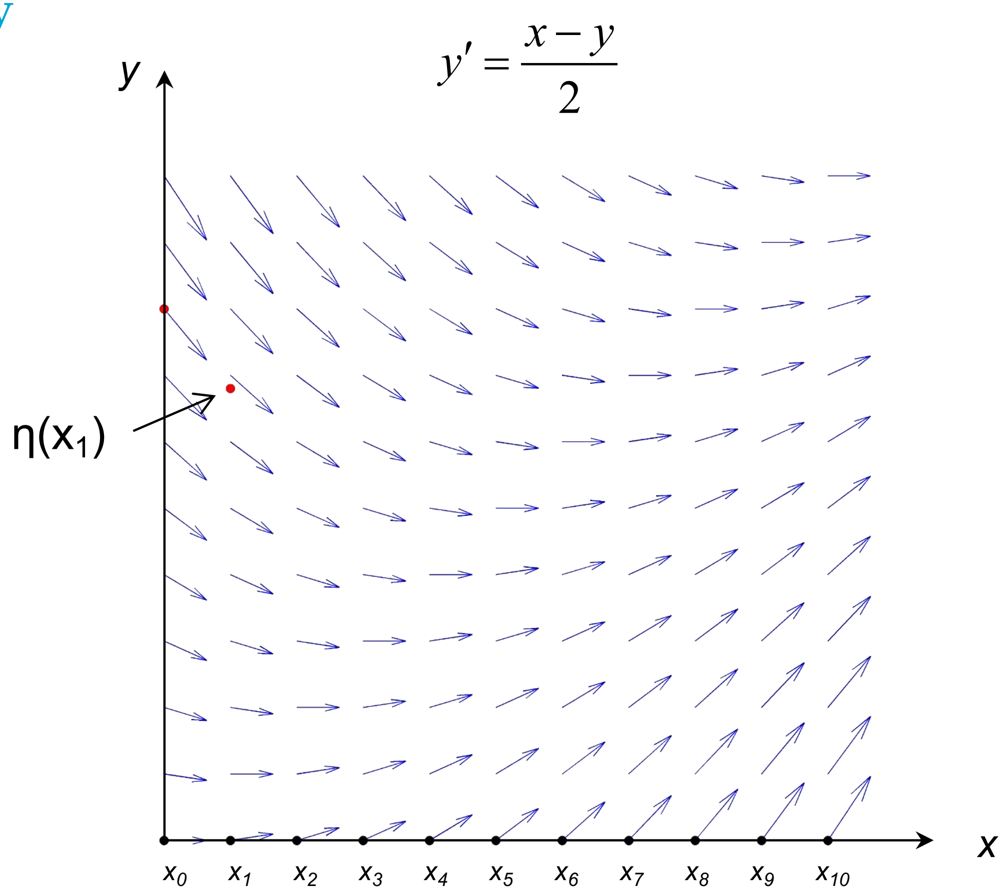
Approximation at $x_1$:
$y(x_1) \approx y(x_0) + y'(x_0)[x_1-x_0]$

$\eta(x_1) = \eta(x_0) + \eta'(x_0)[x_1-x_0]$



# Solving ODEs

## Forward Euler-Cauchy



$$y' = \frac{x-y}{2}$$

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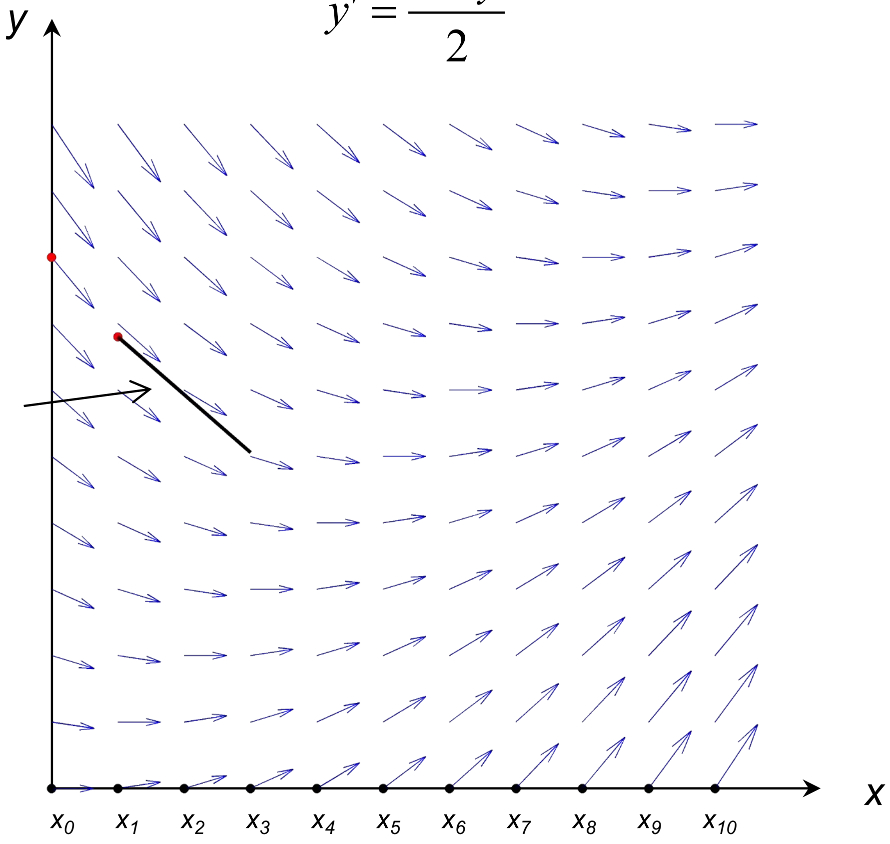
# Solving ODEs

## Forward Euler-Cauchy

$$y' = \frac{x-y}{2}$$



slope $\eta'(x_1)=(x_1-\eta(x_1))/2$

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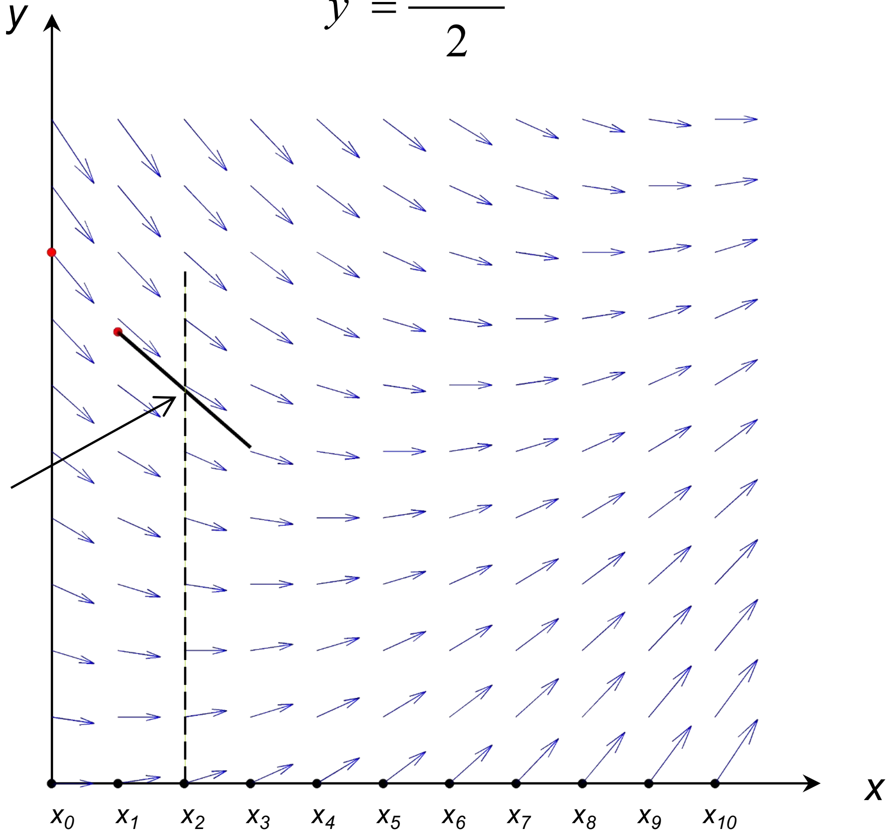
# Solving ODEs

## Forward Euler-Cauchy

$$y' = \frac{x-y}{2}$$



$\eta(x_2) = \eta(x_1) + \eta'(x_1)[x_2-x_1]$

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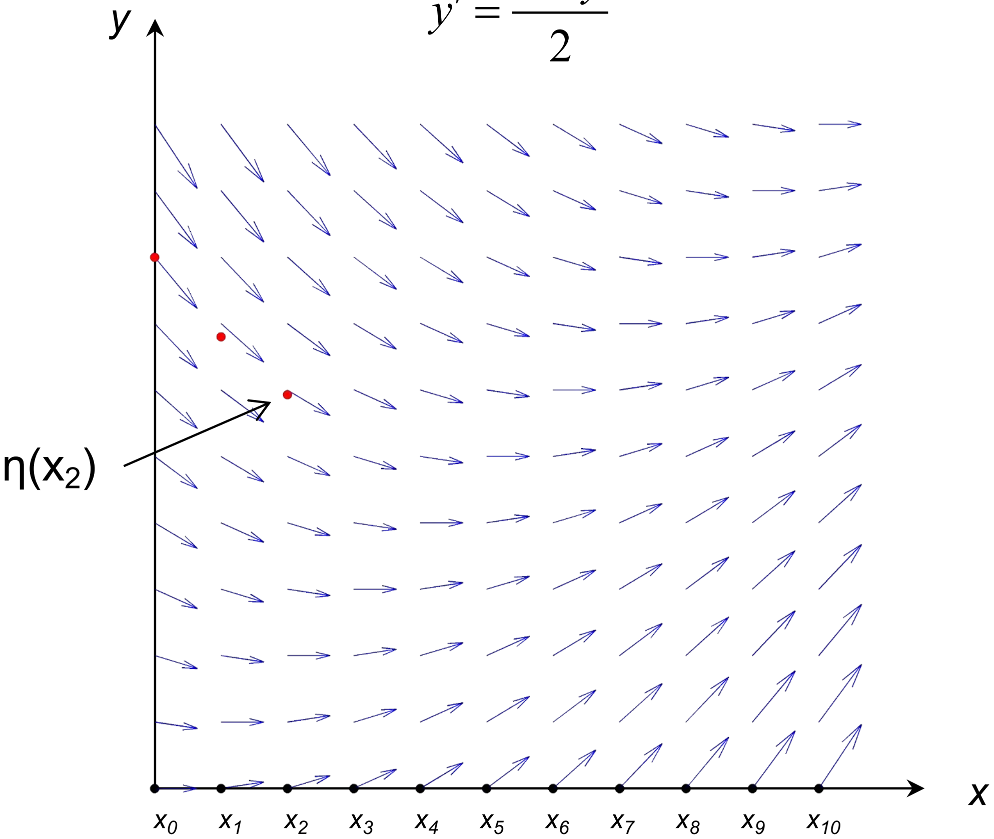
# Solving ODEs

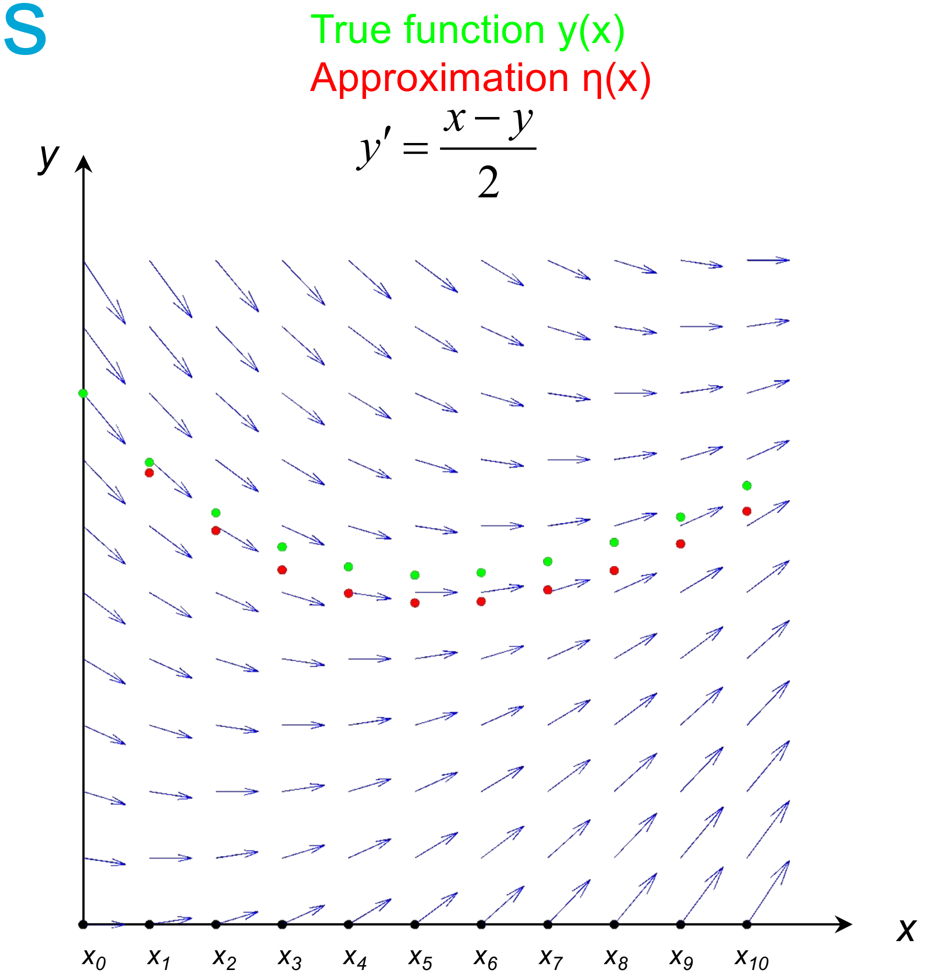
## Forward Euler-Cauchy

$$y' = \frac{x-y}{2}$$



# Solving ODEs

## Forward Euler-Cauchy



$$y' = \frac{x-y}{2}$$

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# Solving ODEs

## Forward Euler-Cauchy

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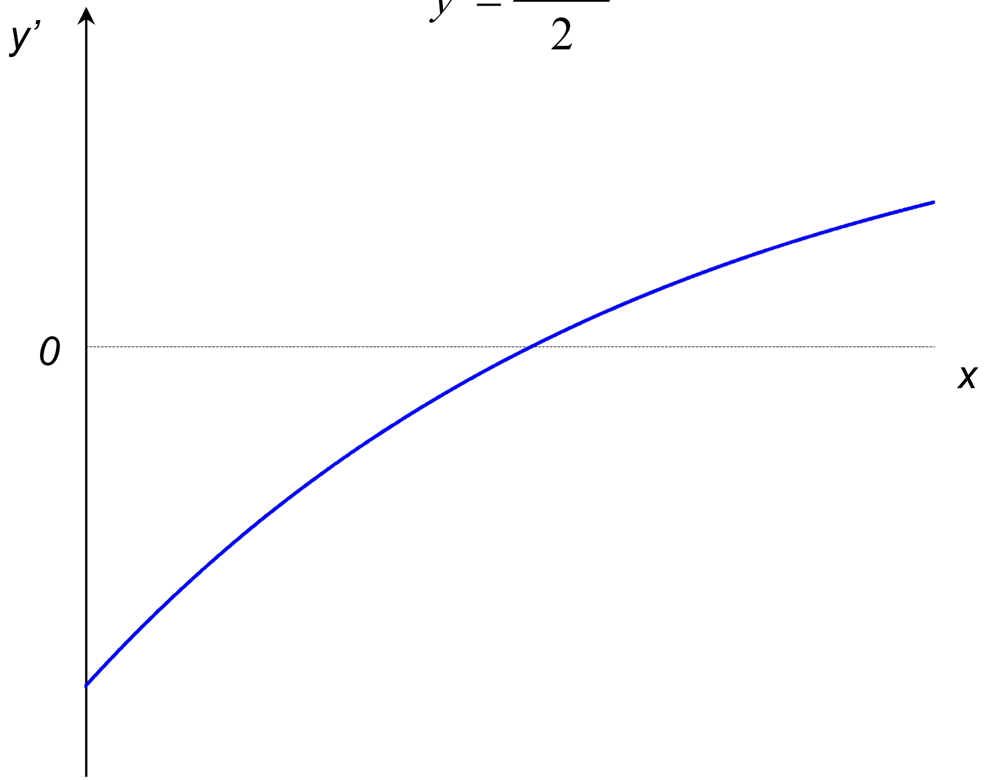

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# Solving ODEs

## Forward Euler-Cauchy

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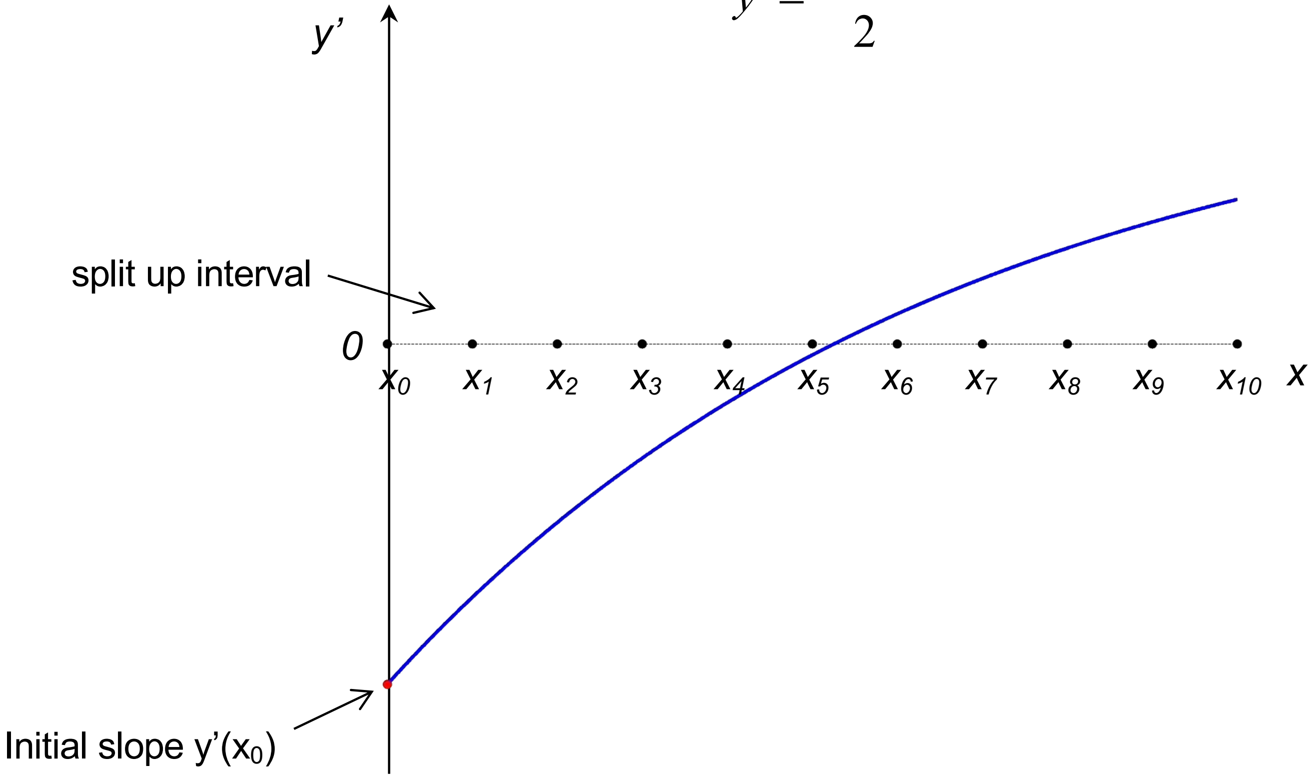
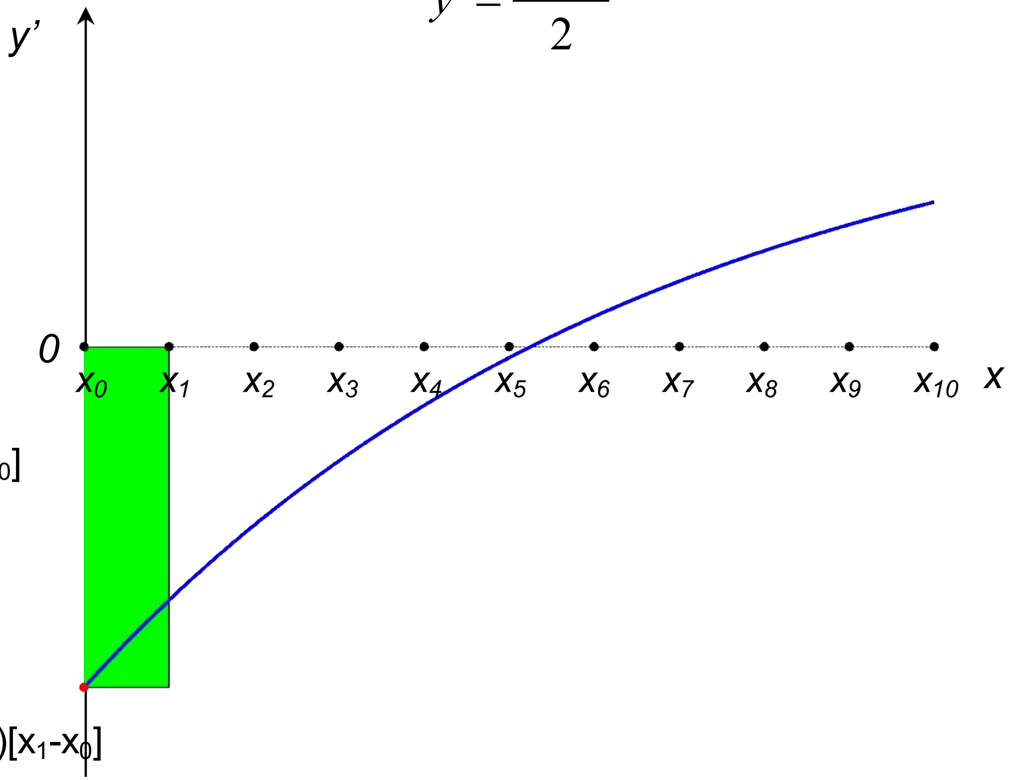


# Solving ODEs

## Forward Euler-Cauchy

$$y' = \frac{x - y}{2}$$



Area = $y'(x_0)[x_1-x_0]$

$+y(x_0)$

$\eta(x_1) = y(x_0) + y'(x_0)[x_1-x_0]$

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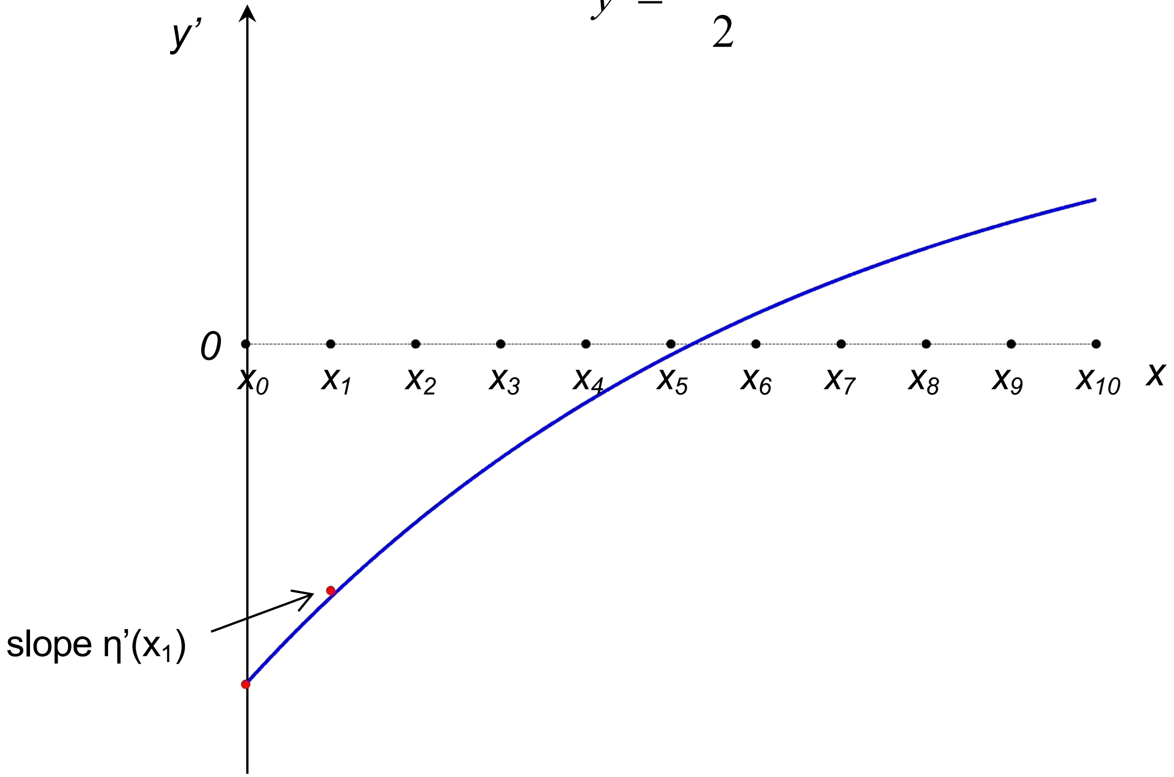
# Solving ODEs

## Forward Euler-Cauchy

$$y' = \frac{x - y}{2}$$



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# Solving ODEs

## Forward Euler-Cauchy

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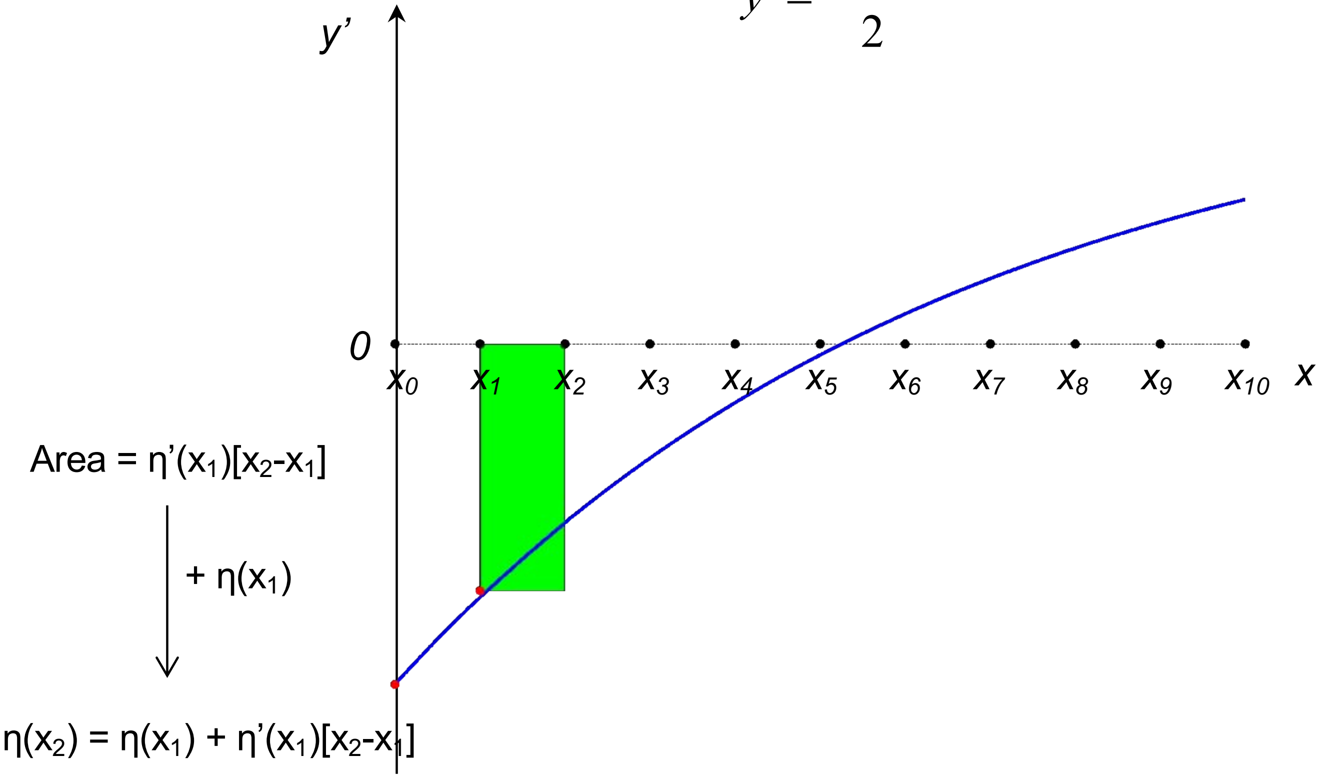

Area = $\eta'(x_1)[x_2-x_1]$

$+ \eta(x_1)$

$\eta(x_2) = \eta(x_1) + \eta'(x_1)[x_2-x_1]$

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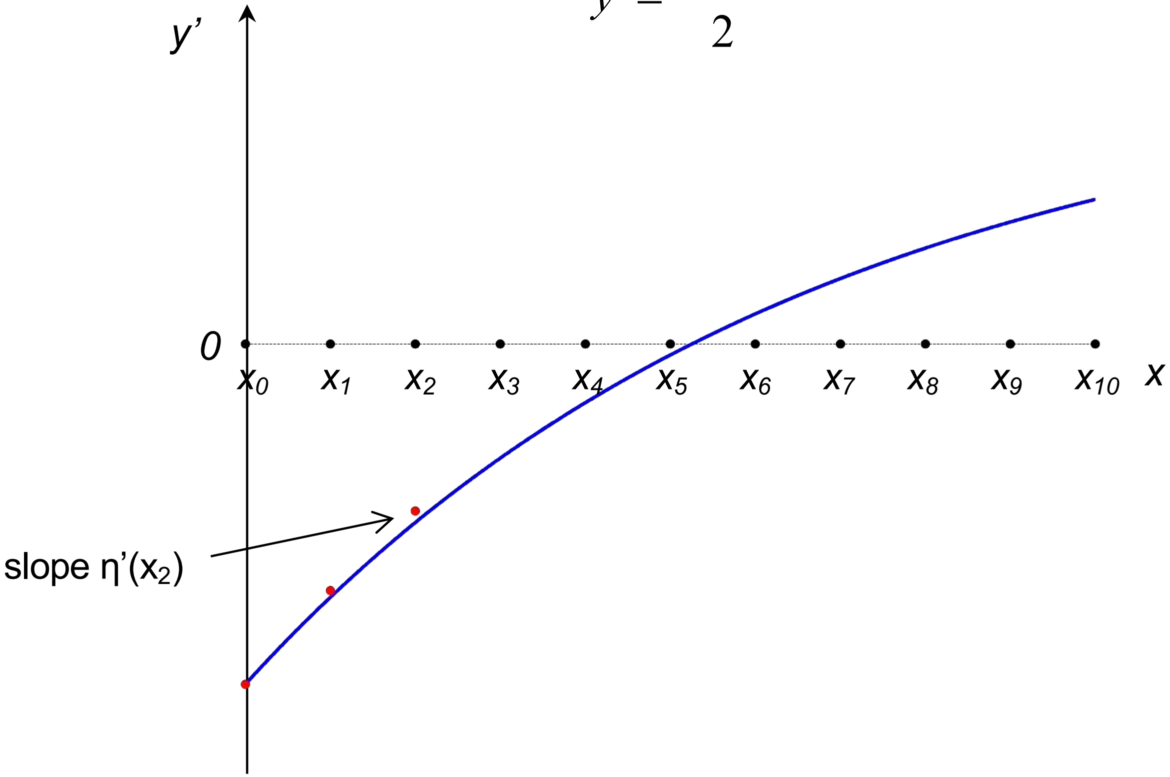
# Solving ODEs

## Forward Euler-Cauchy

$$y' = \frac{x-y}{2}$$



# Solving ODEs

## Forward Euler-Cauchy

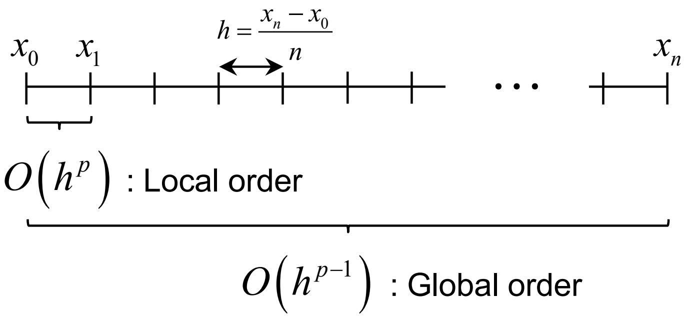
$$\eta_{i+1} = \eta_i + h \cdot f\left(x_i, \eta_i\right) \qquad \eta_0 = y\left(x_0\right)$$

$$y_{i+1} = y_i + h \cdot f\left(x_i, y_i\right) + O\left(h^p\right)$$

- Taylor expansion of $y_{i+1}$

$$y_{i+1} = y_i + \underset{\substack{\downarrow \\ f\left(x_i, y_i\right)}}{\left.\frac{dy}{dx}\right|_i} \cdot h + \underbrace{\frac{1}{2}\left.\frac{d^2y}{dx^2}\right|_i \cdot h^2 + \frac{1}{6}\left.\frac{d^3y}{dx^3}\right|_i \cdot h^3 + O\left(h^4\right)}_{O\left(h^2\right)}$$

Local order of convergence: 2



$O\left(h^p\right)$ : Local order

$O\left(h^{p-1}\right)$ : Global order

**Proof**

$$O\left(n \cdot h^p\right) \equiv O\left(\frac{\left(x_n - x_0\right)}{h} \cdot h^p\right) \equiv O\left(h^{p-1}\right)$$

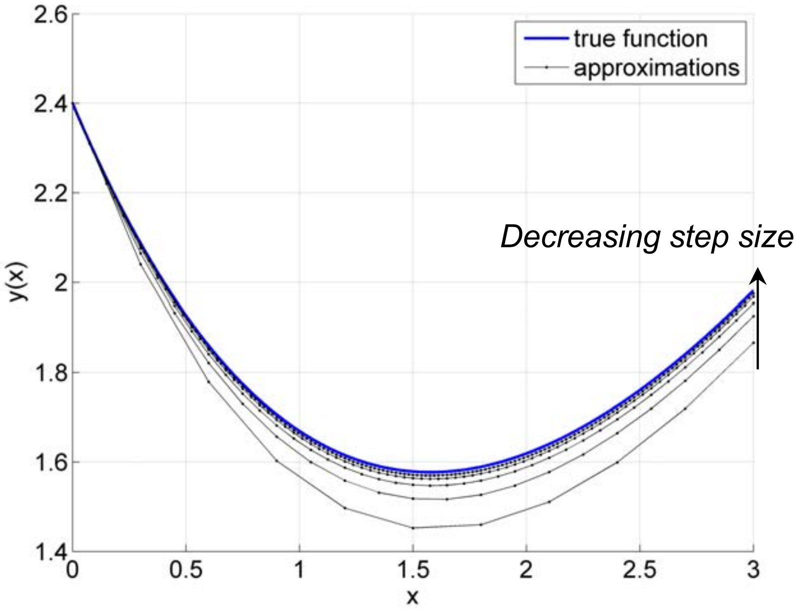
Global order is always one less than local order!!!

# Solving ODEs

## Forward Euler-Cauchy

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_i, \eta_i\right) \qquad \eta_0 = y\left(x_0\right)$$

| h | $\lvert\eta_1 - y_1\rvert$ | $\lvert\eta_n - y_n\rvert$ |
|---|---|---|
| 0.3000 | 0.0471 | 0.1155 |
| 0.1500 | 0.0121 | 0.0565 |
| 0.0750 | 0.0031 | 0.0279 |
| 0.0375 | 0.0008 | 0.0139 |
| 0.0187 | 0.0002 | 0.0069 |



*Decreasing step size*

# Solving ODEs

## Forward Euler-Cauchy

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_i, \eta_i\right) \qquad \eta_0 = y\left(x_0\right)$$

| | h | | $\|\eta_1-y_1\|$ | $\|\eta_n-y_n\|$ | |
|---|---|---|---|---|---|
| | 0.3000 | | 0.0471 | 0.1155 | |
| x 0.5 | | x 0.26 | | | x 0.49 |
| | 0.1500 | | 0.0121 | 0.0565 | |
| x 0.5 | | x 0.25 | | | x 0.49 |
| | 0.0750 | | 0.0031 | 0.0279 | |
| x 0.5 | | x 0.25 | | | x 0.50 |
| | 0.0375 | | 0.0008 | 0.0139 | |
| x 0.5 | | x 0.25 | | | x 0.50 |
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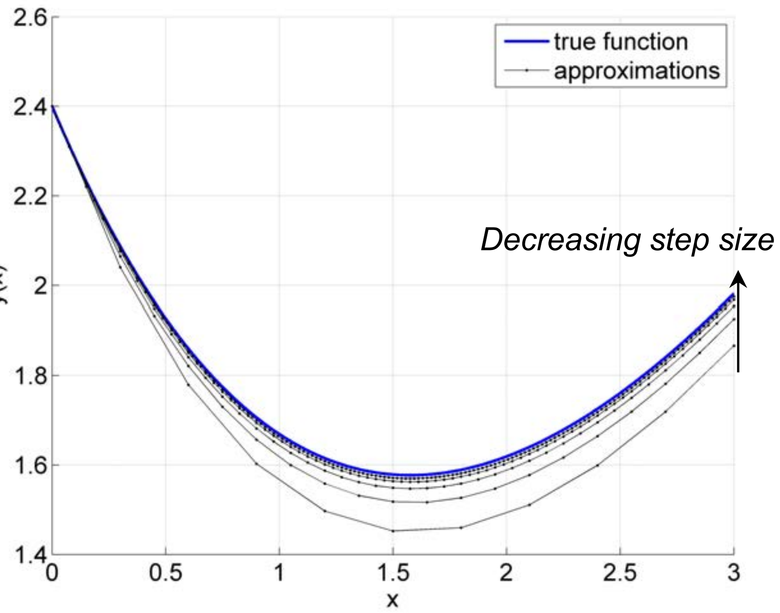

*Local order of convergence: 2*
*Global order of convergence: 1*

# Solving ODEs

## Forward Euler-Cauchy

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_i, \eta_i\right) \qquad \eta_0 = y\left(x_0\right)$$

- Explicit
- Single step
- Local order of convergence: 2
- Global order of convergence: 1

# Solving ODEs

## Backward Euler-Cauchy

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_{i+1}, \eta_{i+1}\right) \qquad \eta_0 = y\left(x_0\right)$$

- Implicit formula: unknown $\eta_{i+1}$ on both sides → recall fixed-point iteration from Module 1

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_{i+1}, \eta_{i+1}\right)$$

$$\eta_{i+1} = \varphi\left(\eta_{i+1}\right)$$

$$\eta_{i+1}^{(k+1)} = \varphi\left(\eta_{i+1}^{(k)}\right) \qquad k = 0, 1, \ldots, k_0$$

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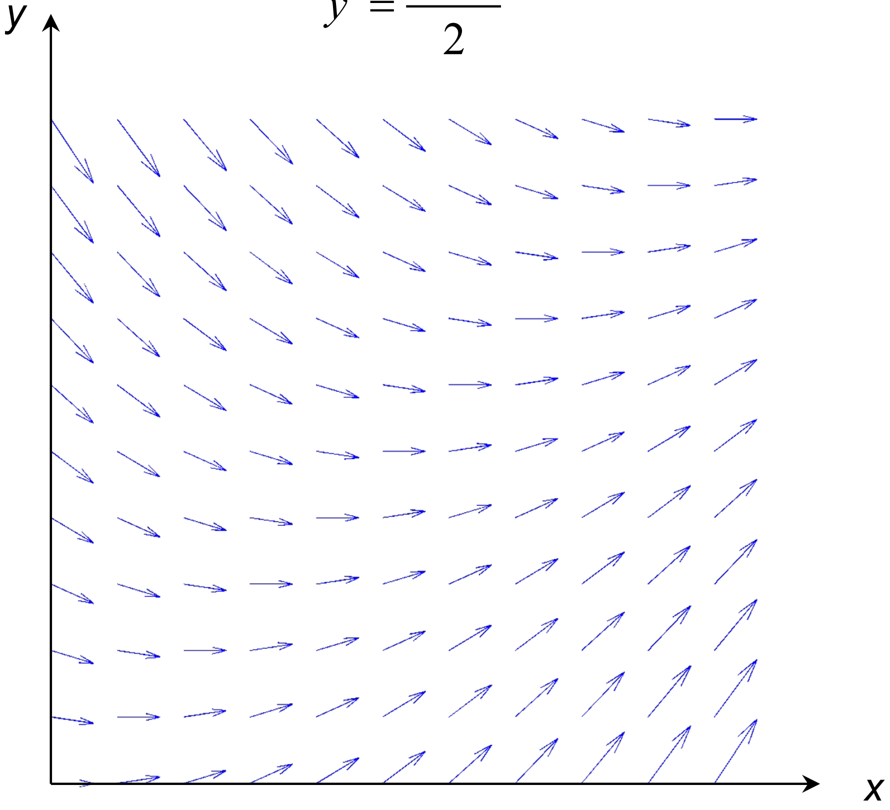
# Solving ODEs

## Backward Euler-Cauchy

$$y(x) = C\exp[-x/2] + x - 2$$

$$y' = \frac{x-y}{2}$$



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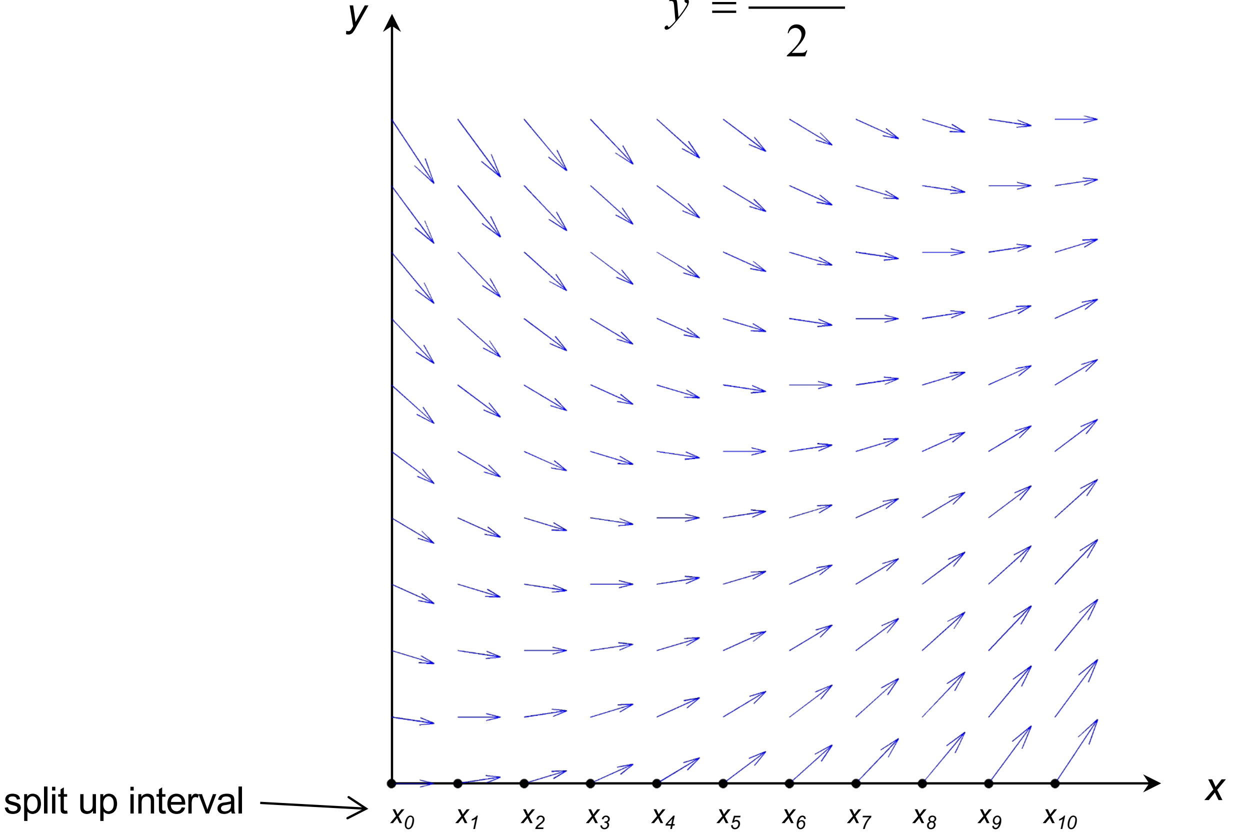
# Solving ODEs

## Backward Euler-Cauchy

$$y' = \frac{x - y}{2}$$



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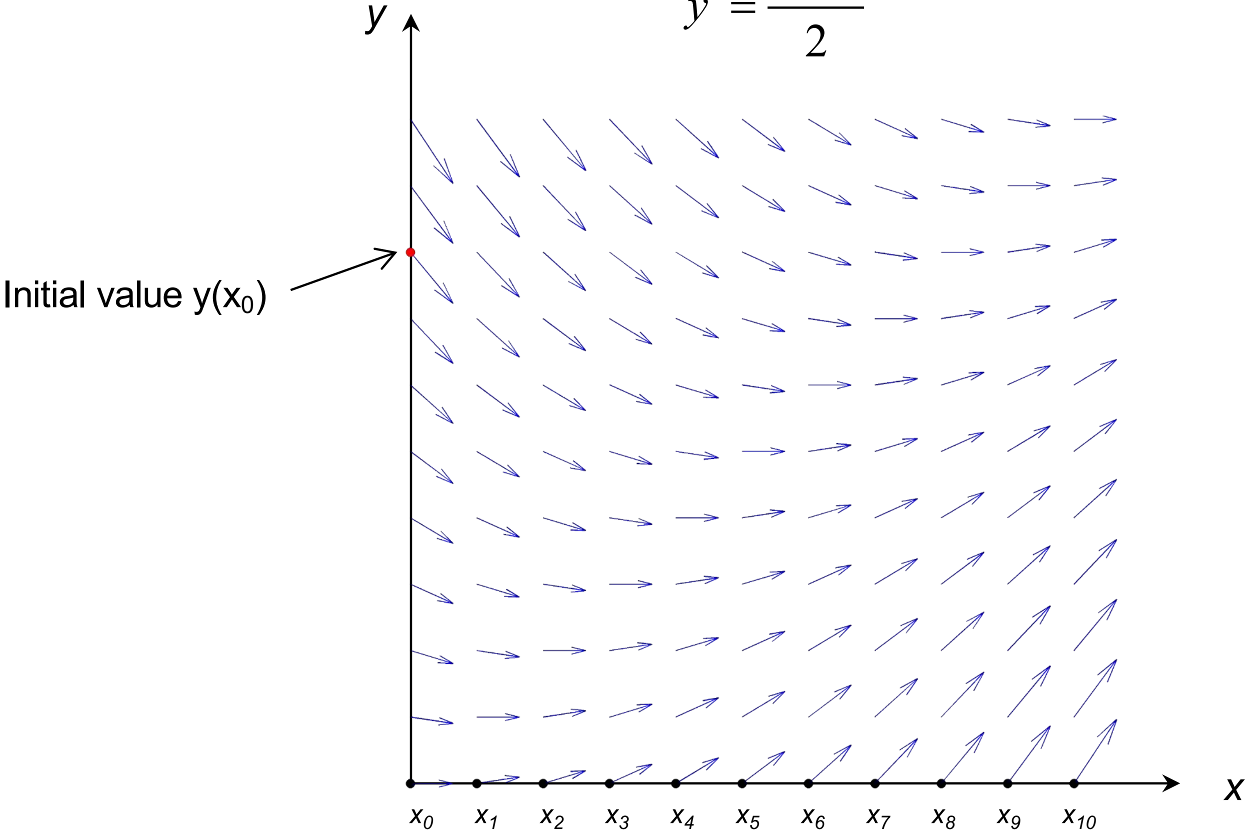
# Solving ODEs

## Backward Euler-Cauchy

$$y(x) = 4.4\exp[-x/2] + x - 2$$

$$y' = \frac{x-y}{2}$$



# Solving ODEs

## Backward Euler-Cauchy

$$y(x) = 4.4\exp[-x/2] + x - 2$$

$$y' = \frac{x-y}{2}$$

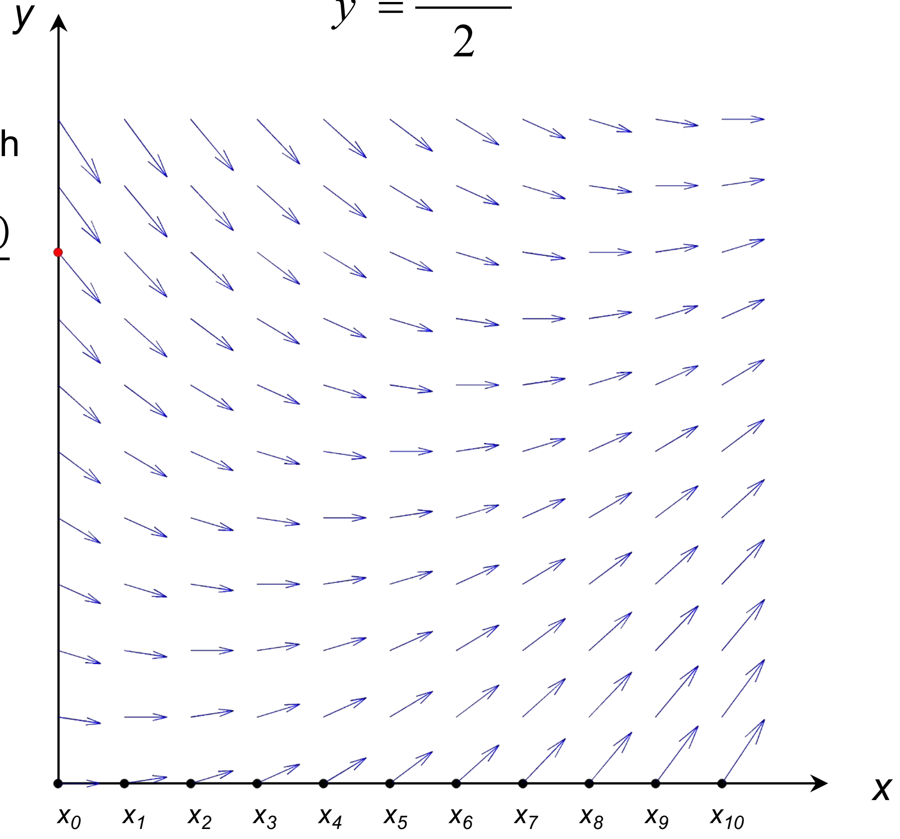
Approximate slope at $x_1$ with backward-difference:

$$y'(x_1) \approx \frac{y(x_1) - y(x_1 - h)}{h}$$

$$\eta_1 = \eta_0 + h \cdot f(x_1, \eta_1)$$

$$\eta_1^{(k+1)} = \varphi\left(\eta_1^{(k)}\right)$$

$$\eta_1^{(0)} = \eta_0 + h \cdot f(x_0, \eta_0)$$



# Solving ODEs

## Backward Euler-Cauchy

$$y' = \frac{x-y}{2}$$

$$\eta_1 = \eta_0 + h \cdot f\left(x_1, \eta_1\right)$$

$$\eta_1^{(k+1)} = \varphi\left(\eta_1^{(k)}\right)$$

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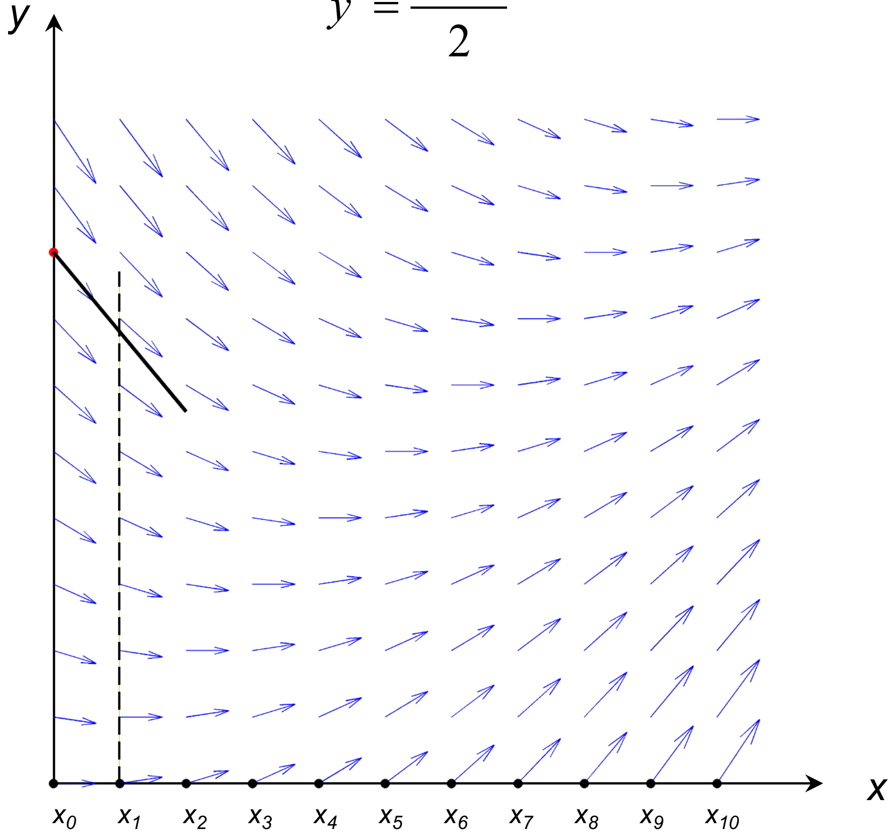
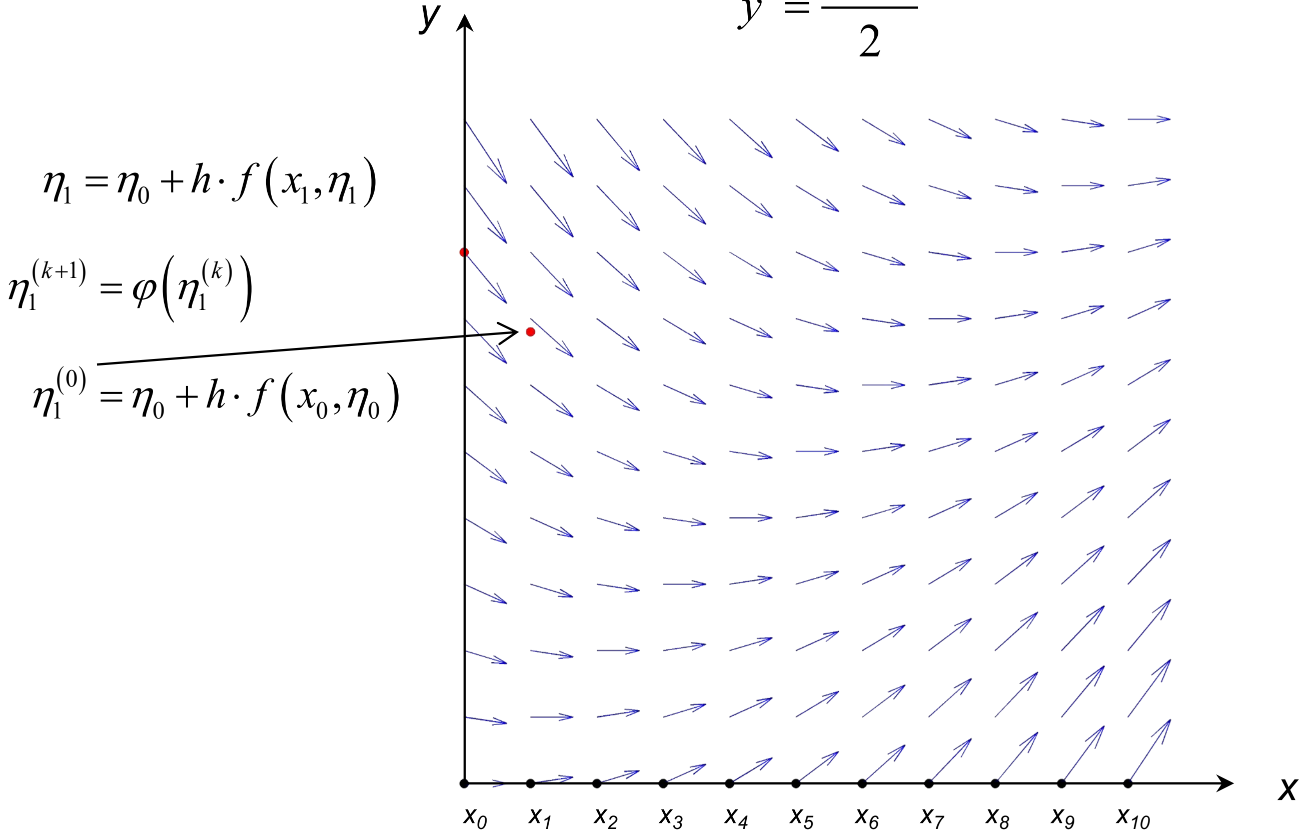


# Solving ODEs

## Backward Euler-Cauchy

$$y' = \frac{x-y}{2}$$



$$\eta_1 = \eta_0 + h \cdot f\left(x_1, \eta_1\right)$$

$$\eta_1^{(k+1)} = \varphi\left(\eta_1^{(k)}\right)$$

$$\eta_1^{(0)} = \eta_0 + h \cdot f\left(x_0, \eta_0\right)$$

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# Solving ODEs

## Backward Euler-Cauchy

$$y' = \frac{x-y}{2}$$

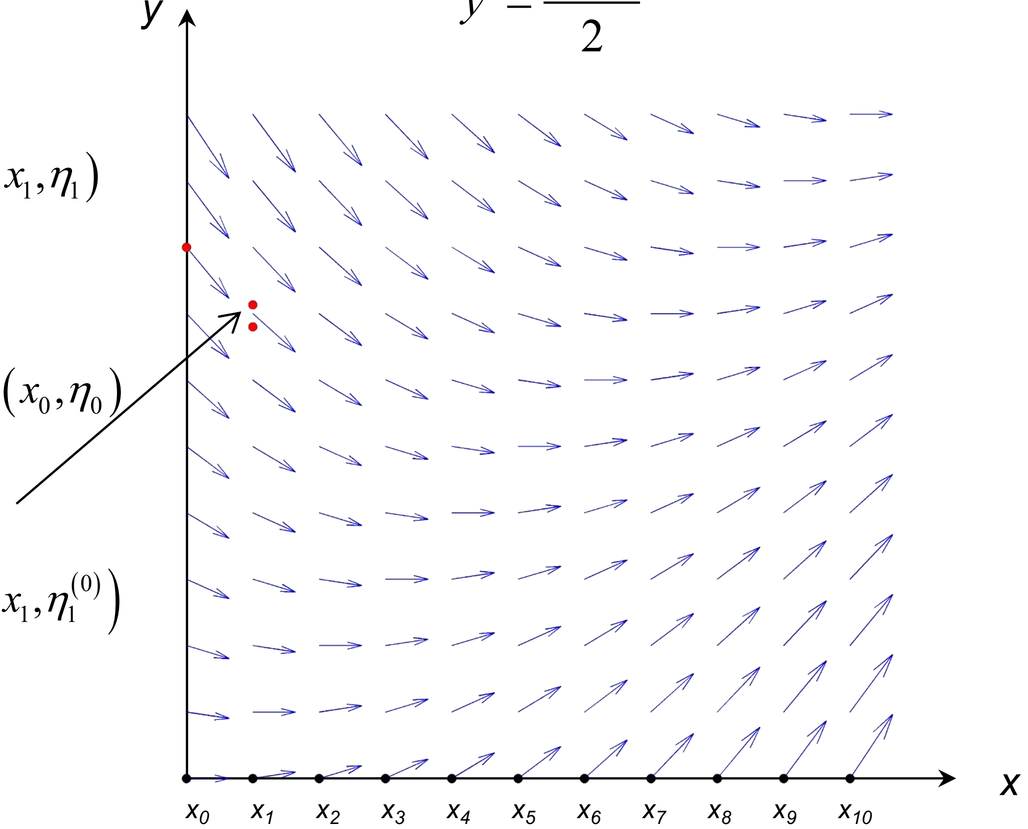
$$\eta_1 = \eta_0 + h \cdot f\left(x_1, \eta_1\right)$$

$$\eta_1^{(k+1)} = \varphi\left(\eta_1^{(k)}\right)$$

$$\eta_1^{(0)} = \eta_0 + h \cdot f\left(x_0, \eta_0\right)$$

$$\eta_1^{(1)} = \varphi\left(\eta_1^{(0)}\right)$$

$$\eta_1^{(1)} = \eta_0 + h \cdot f\left(x_1, \eta_1^{(0)}\right)$$



# Solving ODEs

## Backward Euler-Cauchy

$$y' = \frac{x-y}{2}$$

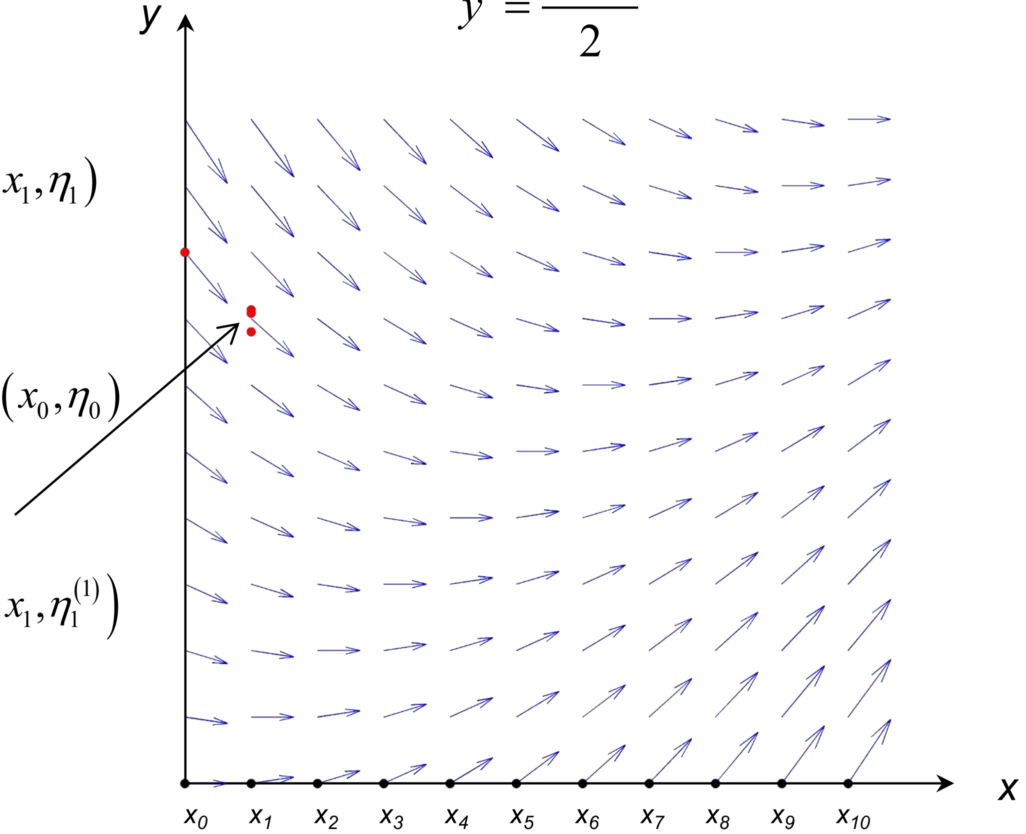
$$\eta_1 = \eta_0 + h \cdot f\left(x_1, \eta_1\right)$$

$$\eta_1^{(k+1)} = \varphi\left(\eta_1^{(k)}\right)$$

$$\eta_1^{(0)} = \eta_0 + h \cdot f\left(x_0, \eta_0\right)$$

$$\eta_1^{(2)} = \varphi\left(\eta_1^{(1)}\right)$$

$$\eta_1^{(2)} = \eta_0 + h \cdot f\left(x_1, \eta_1^{(1)}\right)$$



# Solving ODEs

## Backward Euler-Cauchy

$$y' = \frac{x-y}{2}$$

$$\eta_1 = \eta_0 + h \cdot f\left(x_1, \eta_1\right)$$

$$\eta_1^{(k+1)} = \varphi\left(\eta_1^{(k)}\right)$$

$$\eta_1^{(0)} = \eta_0 + h \cdot f\left(x_0, \eta_0\right)$$

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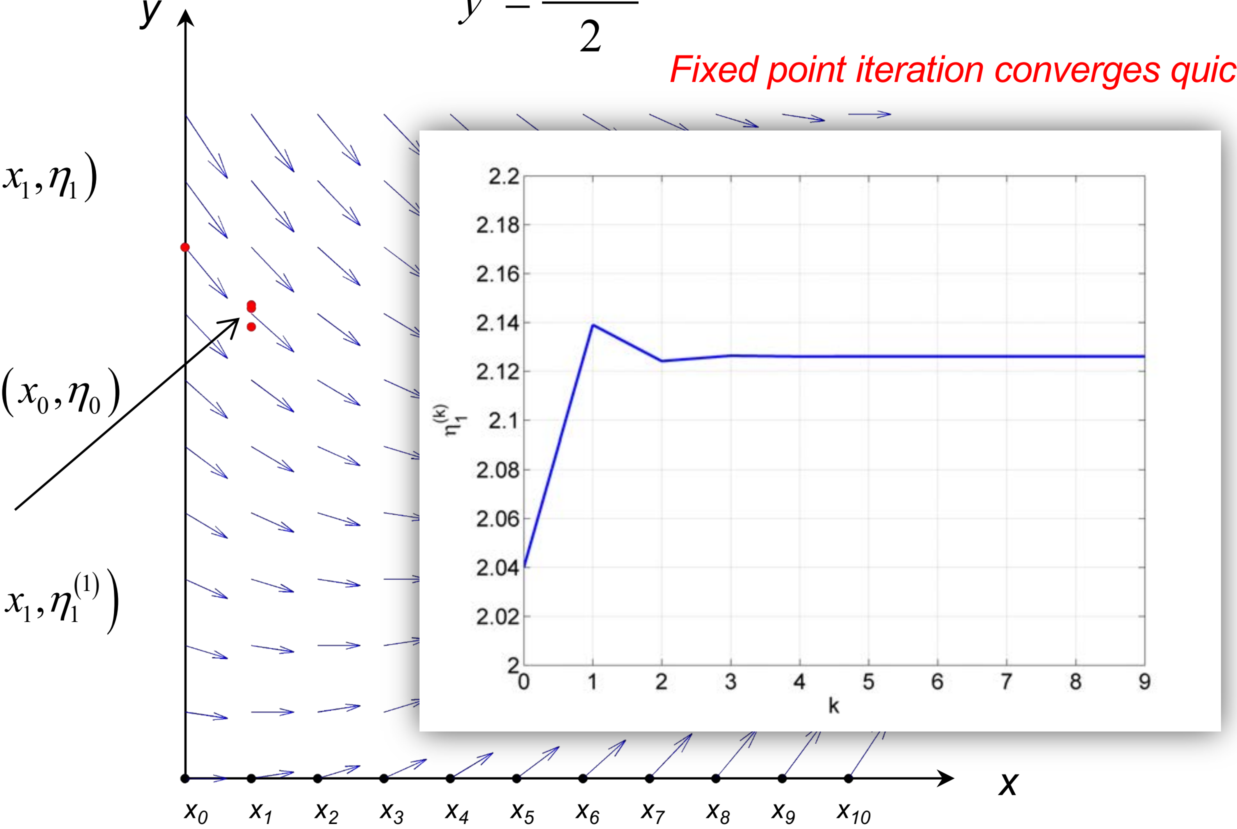


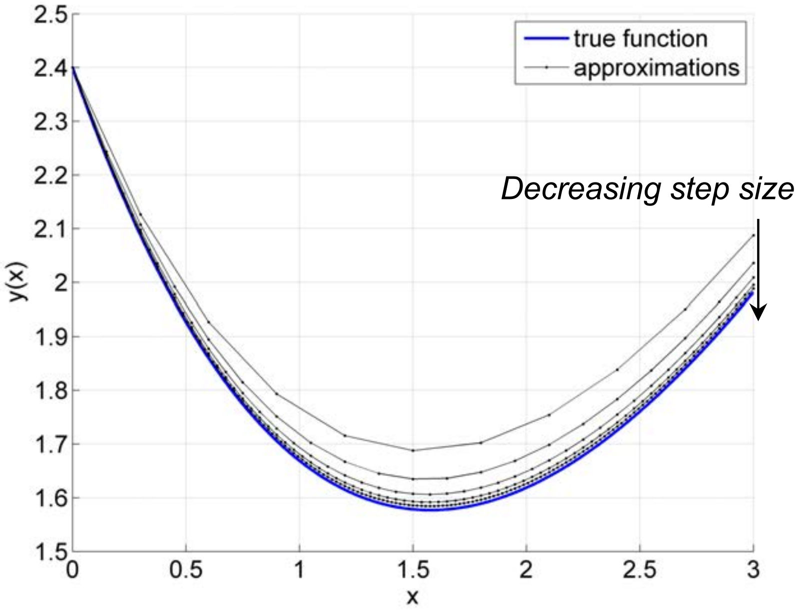
*Fixed point iteration converges quickly*

# Solving ODEs

## Backward Euler-Cauchy

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_{i+1}, \eta_{i+1}\right) \qquad \eta_0 = y\left(x_0\right)$$

| h | $\lvert\eta_1 - y_1\rvert$ | $\lvert\eta_n - y_n\rvert$ |
|---|---|---|
| 0.3000 | 0.0390 | 0.1058 |
| 0.1500 | 0.0110 | 0.0540 |
| 0.0750 | 0.0029 | 0.0273 |
| 0.0375 | 0.0007 | 0.0137 |
| 0.0187 | 0.0002 | 0.0069 |



Decreasing step size

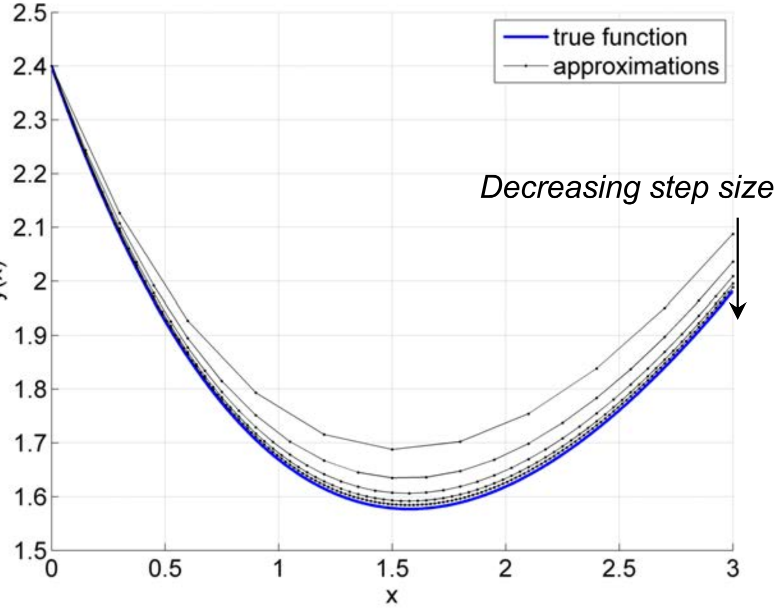
# Solving ODEs

## Backward Euler-Cauchy

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_{i+1}, \eta_{i+1}\right) \qquad \eta_0 = y\left(x_0\right)$$

| | h | | $\lvert\eta_1 - y_1\rvert$ | $\lvert\eta_n - y_n\rvert$ | |
|---|---|---|---|---|---|
| | 0.3000 | | 0.0390 | 0.1058 | |
| x 0.5 | | x 0.28 | | | x 0.51 |
| | 0.1500 | | 0.0110 | 0.0540 | |
| x 0.5 | | x 0.27 | | | x 0.51 |
| | 0.0750 | | 0.0029 | 0.0273 | |
| x 0.5 | | x 0.26 | | | x 0.50 |
| | 0.0375 | | 0.0007 | 0.0137 | |
| x 0.5 | | x 0.25 | | | x 0.50 |
| | 0.0187 | | 0.0002 | 0.0069 | |

*Local order of convergence: 2*
*Global order of convergence: 1*

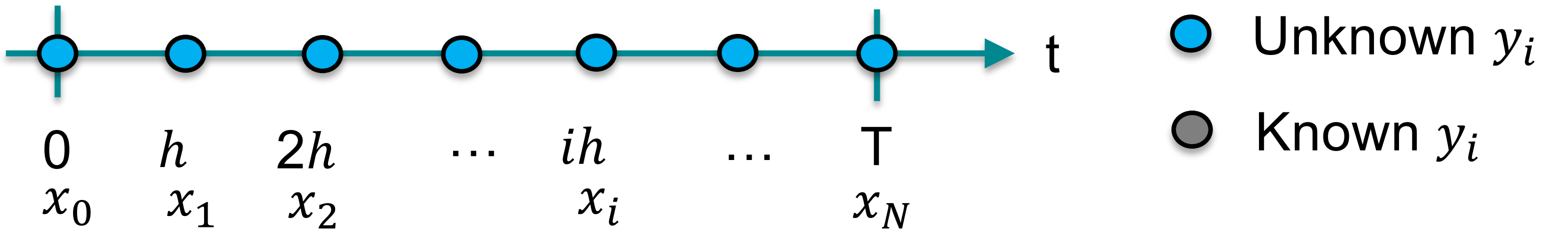


*Decreasing step size*

# Solution marching: forward vs backward differencing

Governing equation : $y'(x) = f(y)$
Initial condition : $y(0) = y_0$

Solution on the interval [0,T], approximated by $y_i = y(x_0 + ih)$



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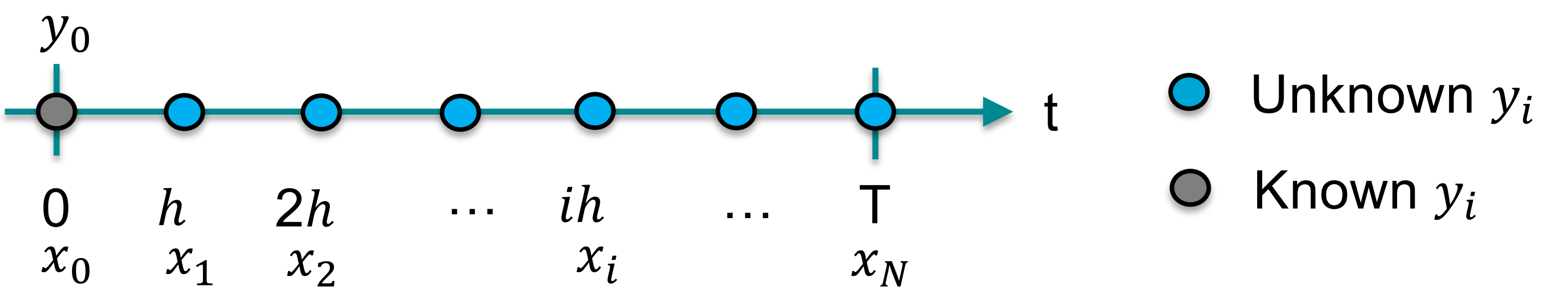
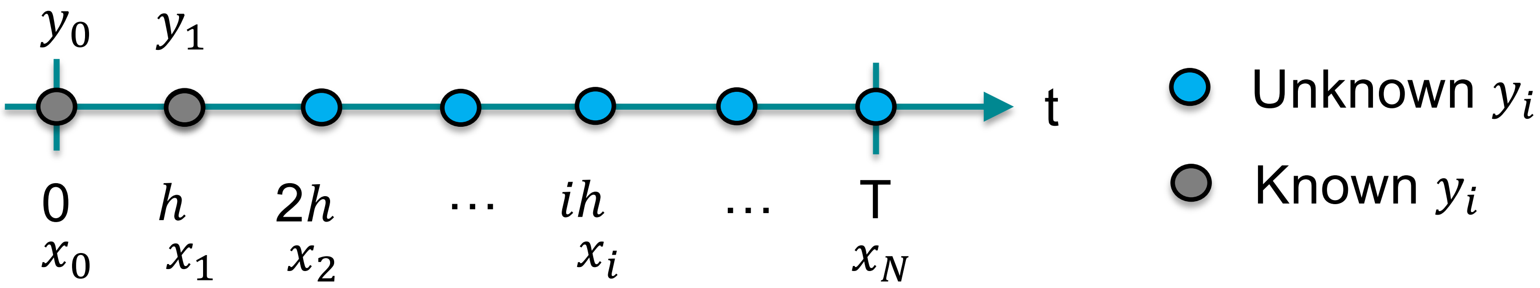


Initial condition : $y_0$ is known!

# Solution marching: forward vs backward differencing

Governing equation : $y'(x) = f(y)$
Initial condition : $y(0) = y_0$

Solution on the interval [0,T], approximated by $y_i = y(x_0 + ih)$



Governing equation: $y'(x_0) = f(x_0)$

Approximate *y'* using a forward difference at $x_0$ :

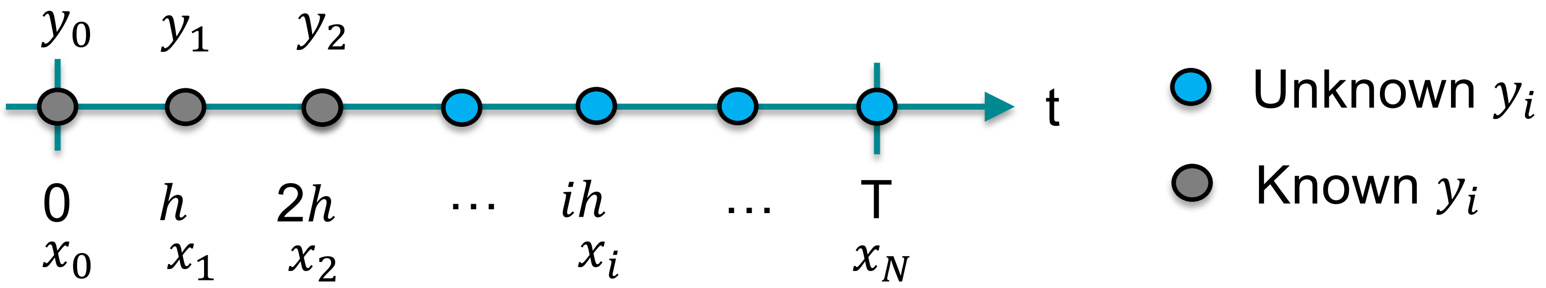
$$\frac{y(x_0+h) - y(x_0)}{h} \approx y'(x_0) = f(x_0) \quad \Longrightarrow y(x_0+h) = y(x_0) + hf(y_0)$$

$$\Longrightarrow y_1 = y_0 + hf(y_0)$$

# Solution marching: forward vs backward differencing

Governing equation : $y'(x) = f(y)$
Initial condition : $y(0) = y_0$

Solution on the interval [0,T], approximated by $y_i = y(x_0 + ih)$



Governing equation: $y'(x_1) = f(y_1)$

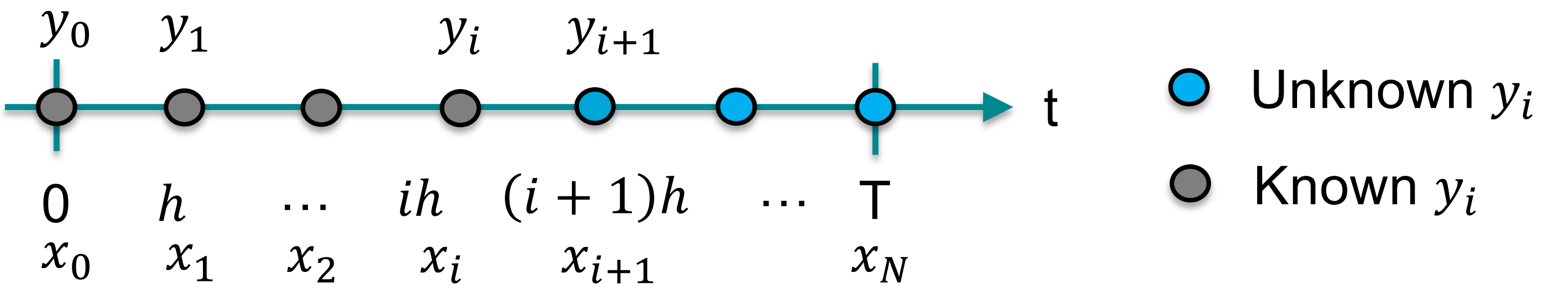
Approximate *y’* using a forward difference at *x₁* :

$$\frac{y(x_1+h) - y(x_1)}{h} \approx y'(x_1) = f(y_1) \qquad \begin{aligned} &\Rightarrow y(x_1+h) = y(x_1) + hf(y_1) \\ &\Rightarrow y_2 = y_1 + hf(y_1) \end{aligned}$$

# Solution marching: forward vs backward differencing

Governing equation : $y'(x) = f(y)$
Initial condition : $y(0) = y_0$

Solution on the interval [0,T], approximated by $y_i = y(x_0 + ih)$



Governing equation: $y'(t_i) = f(y_i)$

Approximate *y'* using a forward difference at $x_i$ :

$$\frac{y(x_i + h) - y(t_i)}{h} \approx y'(x_i) = f(y_i)$$

$$\Rightarrow y(x_i + h) = y(x_i) + hf(y_i)$$

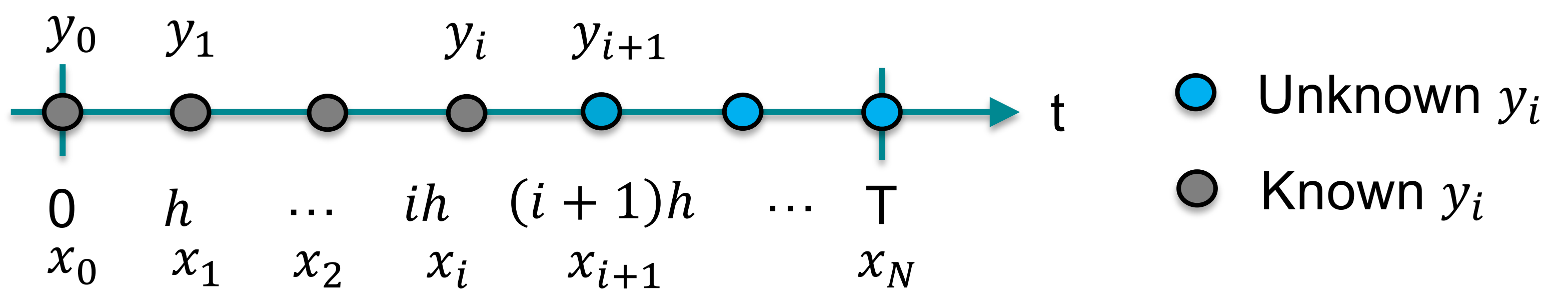
$$\boxed{\Rightarrow y_{i+1} = y_i + hf(y_i)}$$

# Solution marching: forward vs backward differencing

Governing equation : $y'(x) = f(y)$

Initial condition : $y(0) = y_0$

Solution on the interval [0,T], approximated by $y_i = y(x_0 + ih)$



Governing equation: $y'(x_i + h) = f(y_{i+1})$

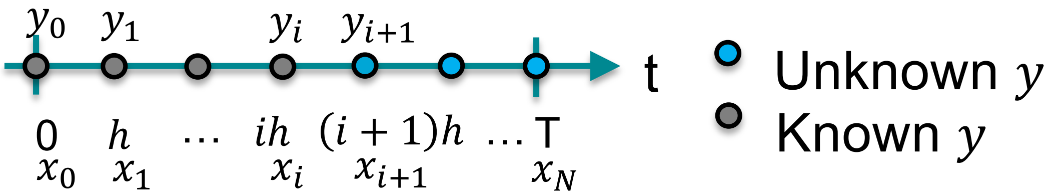
Approximate *y'* using a ***backward*** difference at $t_{i+1}$ :

$$\frac{y(x_i + h) - y(x_i)}{h} \approx y'(x_{i+1}) = f(y_{i+1}) \quad \boxed{\Longrightarrow y_{i+1} = y_i + hf(y_{i+1})}$$

# Forward vs Backward Euler

Governing equation:

$$y'(x) = f(y)$$



**Forward (explicit) Euler**

- Uses forward difference for y’
- Global truncation error $O(h)$
- Explicit formulation for $y_{i+1}$
- Local truncation error $O(h^2)$
- “Cheap” to solve

$$\boxed{y'(x_i) = \frac{y(x_i + h) - y(x_i)}{h} + O(h)}$$

$$\boxed{y_{i+1} = y_i + hf(y_i) + O(h^2)}$$

**Backward (implicit) Euler**

- Uses backward difference for y’
- Global truncation error $O(h)$
- Implicit formulation for $y_{i+1}$
- Local truncation error $O(h^2)$
- “Expensive” to solve

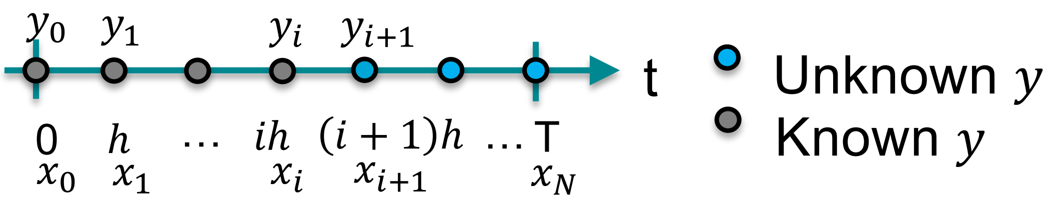
$$\boxed{y'(x_{i+1}) = \frac{y(x_i + h) - y(x_i)}{h} + O(h)}$$

$$\boxed{y_{i+1} = y_i + hf(y_{i+1}) + O(h^2)}$$

# Forward vs Backward Euler

Governing equation:

$$y'(x) = f(y)$$



- ○ Unknown $y$
- ● Known $y$

**Forward (explicit) Euler**

- Uses forward difference for y'

$$y'(x_i) = \frac{y(x_i + h) - y(x_i)}{} + O(h)$$

# Why bother with backward Euler?

**Ba**

$$y'(x_{i+1}) = \frac{}{h} + O(h)$$

- Global truncation error $O(h)$
- Implicit formulation for $y_{i+1}$
- Local truncation error $O(h^2)$
- “Expensive” to solve

$$y_{i+1} = y_i + hf(y_{i+1}) + O(h^2)$$

# Stability

- In practice, discretization errors and rounding errors accumulate during time stepping

- Stable algorithm: error in one step not amplified when performing next steps.

- Unstable algorithm: for arbitrarily large number of steps, difference between approximation and true solution continuously increases.

- Cause instability
  - The ordinary differential equation
    - Stable ODE → stable solution if integration algorithm stable
    - Unstable ODE → no stable solution
  - The integration algorithm

# Stability

## Ordinary differential equation

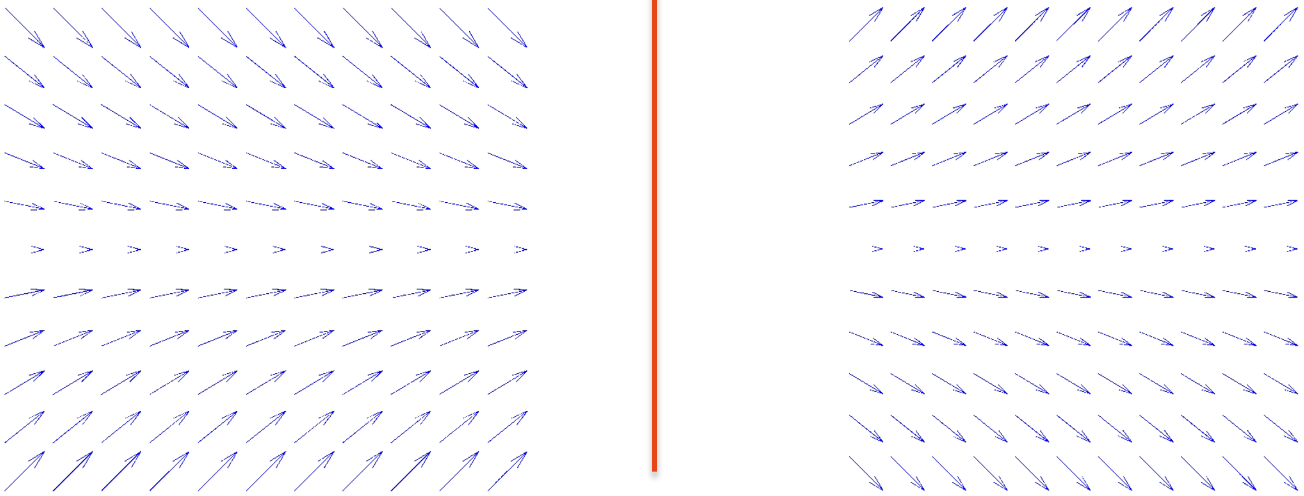
$$y' = cy$$

$$c < 0$$

$$y' = -y$$

$$c > 0$$

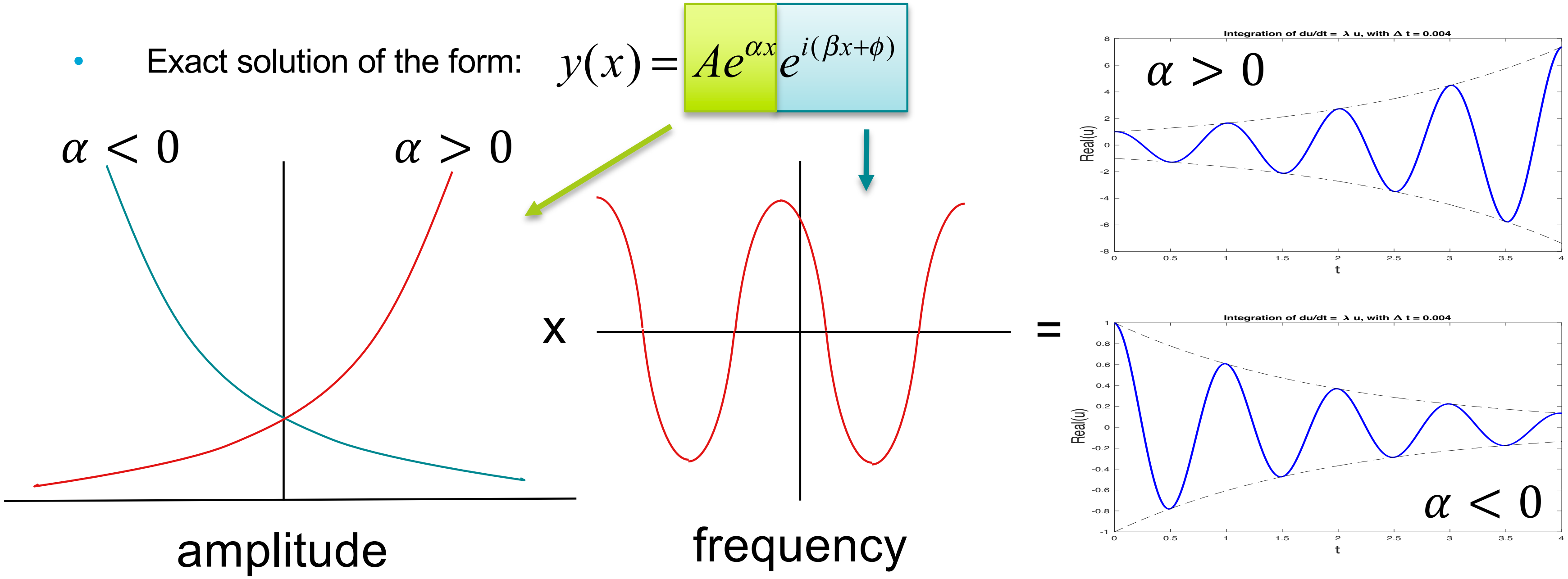
$$y' = y$$



# Stability

## Integration algorithm

- Take as test function: $y' = cy \quad c \in \mathbb{C}$ (c complex)
- $c = \alpha + \beta i$
- Exact solution of the form: $y(x) = \boxed{Ae^{\alpha x}}\boxed{e^{i(\beta x+\phi)}}$



# Stability

## Integration algorithm

- Take as test function: $y' = cy$ $c \in \mathbb{C}$ (c complex)
- We want the numerical integration to be bounded (stable) for cases where $Re(c) \leq 0$: an initial error should not amplify!
- Set up time integration formula for test function
  - Forward Euler Cauchy

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_i, \eta_i\right)$$

$$\eta_{i+1} = \eta_i + h \cdot c \cdot \eta_i = \left(1 + h \cdot c\right)\eta_i$$

$$\eta_{i+1} = \underbrace{\left(1 + h \cdot c\right)}_{\left|1 + h \cdot c\right| \leq 1}{}^{i+1} \eta_0$$

To ensure that errors aren't amplified

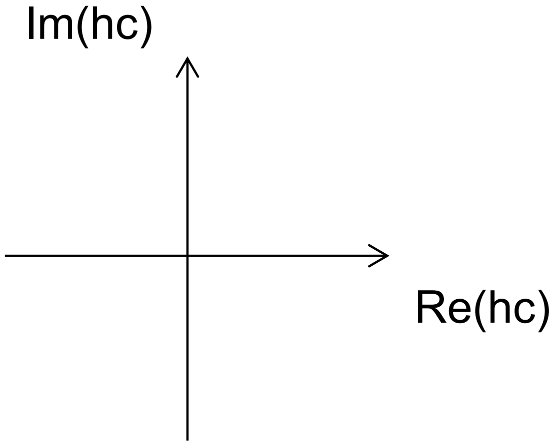
# Stability

## Integration algorithm: Forward Euler Cauchy

$$\left|1+h\cdot c\right|\leq 1$$

- Note that *c* is defined in the complex plane (imaginary values lead to oscillatory solutions of the ODE)

$$c=\alpha+\beta i$$



$$\left|1+h\cdot c\right|=\left|1+h\alpha+h\beta i\right|=\sqrt{\left(1+h\alpha\right)^2+\left(h\beta\right)^2}\leq 1$$

$$\left(1+h\alpha\right)^2+\left(h\beta\right)^2\leq 1$$

# Stability

## Integration algorithm: Forward Euler Cauchy

$$\left|1+h\cdot c\right|\leq 1$$

- Note that *c* is defined in the complex plane (imaginary values lead to oscillatory solutions of the ODE)

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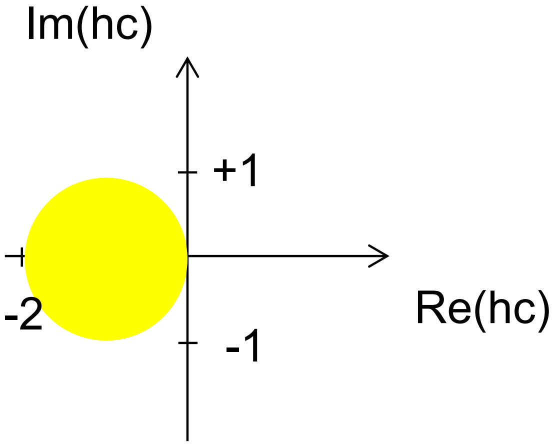


$$\left|1+h\cdot c\right|=\left|1+h\alpha+h\beta i\right|=\sqrt{\left(1+h\alpha\right)^2+\left(h\beta\right)^2}\leq 1$$

$$\left(1+h\alpha\right)^2+\left(h\beta\right)^2\leq 1$$

*Stable when hc inside yellow circle: so method is conditionally stable*

# Stability

## Integration algorithm: Forward Euler Cauchy

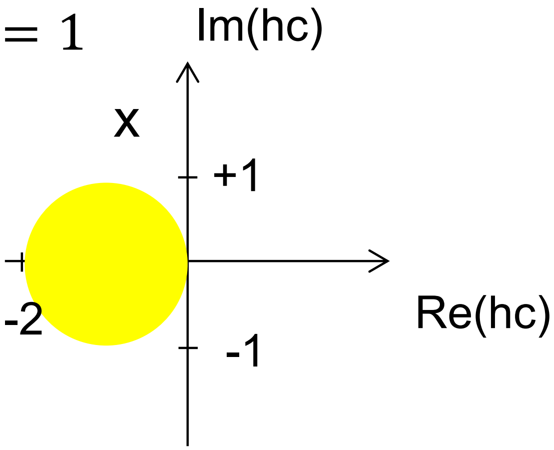
$$\left|1+h\cdot c\right|\leq 1$$

- Note that *c* is defined in the complex plane (imaginary values lead to oscillatory solutions of the ODE)

$$c=\alpha+\beta i$$

$$c=-0.5+1.5i$$

$$h=1$$



$$\left|1+h\cdot c\right|=\left|1+h\alpha+h\beta i\right|=\sqrt{\left(1+h\alpha\right)^2+\left(h\beta\right)^2}\leq 1$$

$$\left(1+h\alpha\right)^2+\left(h\beta\right)^2\leq 1$$

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Ch. 8
Ordinary
Differential
Equations

# Stability

## Integration algorithm: Forward Euler Cauchy

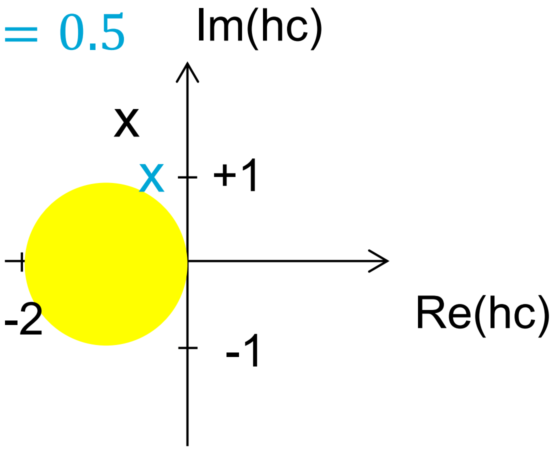
$$\left|1+h\cdot c\right|\leq 1$$

- Note that *c* is defined in the complex plane (imaginary values lead to oscillatory solutions of the ODE)

$$c=\alpha+\beta i$$

$$c = -0.5 + 1.5i$$

$$h = 0.5$$



$$\left|1+h\cdot c\right|=\left|1+h\alpha+h\beta i\right|=\sqrt{\left(1+h\alpha\right)^2+\left(h\beta\right)^2}\leq 1$$

$$\left(1+h\alpha\right)^2+\left(h\beta\right)^2\leq 1$$

*Stable when hc inside yellow circle: so method is conditionally stable*

# Stability

## Integration algorithm: Forward Euler Cauchy

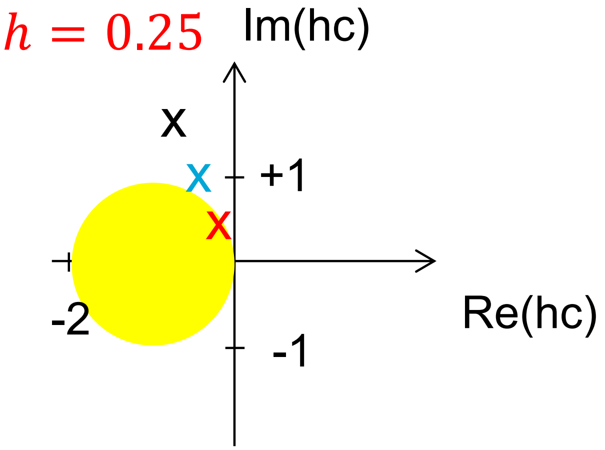
$$|1+h\cdot c|\leq 1$$

- Note that *c* is defined in the complex plane (imaginary values lead to oscillatory solutions of the ODE)

$$c=\alpha+\beta i$$

$c = -0.5 + 1.5i$

$h = 0.25$



$$|1+h\cdot c|=|1+h\alpha+h\beta i|=\sqrt{(1+h\alpha)^2+(h\beta)^2}\leq 1$$

$$(1+h\alpha)^2+(h\beta)^2\leq 1$$

*Stable when hc inside yellow circle: so method is conditionally stable*

Ch. 8
Ordinary
Differential
Equations

# Stability

## Integration algorithm: Forward Euler Cauchy

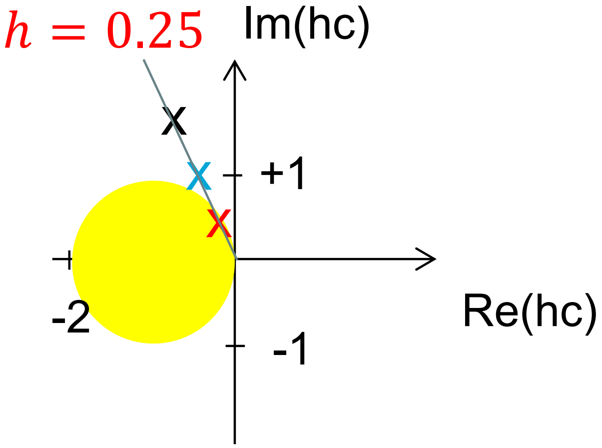
$$\left|1+h\cdot c\right|\leq 1$$

- Note that *c* is defined in the complex plane (imaginary values lead to oscillatory solutions of the ODE)

$$c=\alpha+\beta i$$

$$c=-0.5+1.5i$$

$$h=0.25$$



$$\left|1+h\cdot c\right|=\left|1+h\alpha+h\beta i\right|=\sqrt{\left(1+h\alpha\right)^2+\left(h\beta\right)^2}\leq 1$$

$$\left(1+h\alpha\right)^2+\left(h\beta\right)^2\leq 1$$

*Stable when hc inside yellow circle: so method is conditionally stable*

# Stability

## Integration algorithm

- Take as test function: $y' = cy$
- Set up time integration formula for test function
  - Backward Euler Cauchy

$$\eta_{i+1} = \eta_i + h \cdot f\left(x_{i+1}, \eta_{i+1}\right)$$

$$\eta_{i+1} = \eta_i + h \cdot c \cdot \eta_{i+1}$$

$$\eta_{i+1} = \frac{\eta_0}{\underbrace{\left(1 - h \cdot c\right)^{i+1}}_{\left|\frac{1}{1-h\cdot c}\right| \leq 1}}$$

To ensure that errors aren't amplified

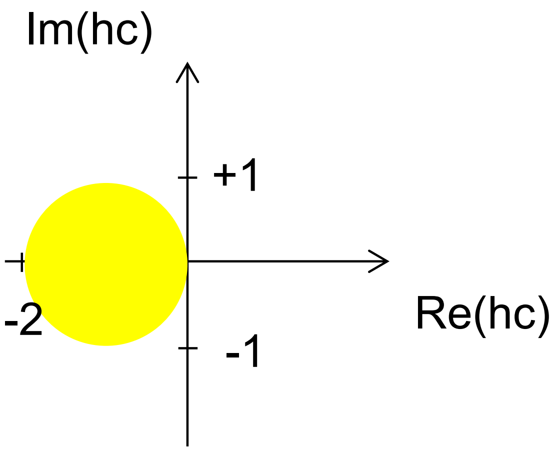
# Stability

We only consider the left hand side of the plane Re(hc)
→ this is where the ODE is stable

## Integration algorithm: Forward vs. Backward Euler Cauchy

Forward Euler Cauchy

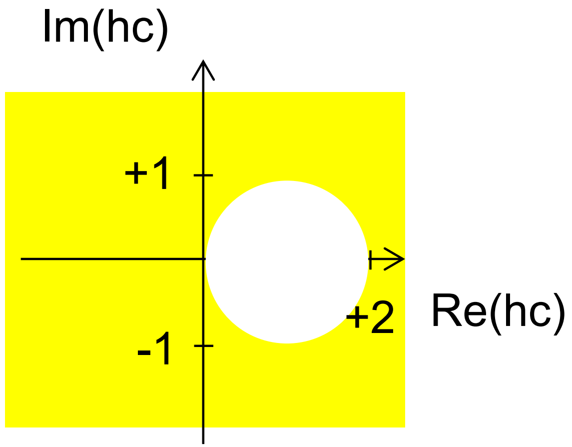
$$|1+h\cdot c|\leq 1$$



Conditionally stable

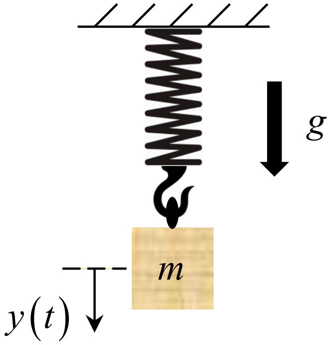
Backward Euler Cauchy

$$\left|\frac{1}{1-h\cdot c}\right|\leq 1$$



Unconditionally stable

# Ordinary differential equations



- Newton's second law of motion: $my''(t) = mg - ky(t)$
- 2nd order ordinary differential equation
- Introduce the auxiliary functions: $z_1(t) = y(t)$, $z_2(t) = y'(t)$
- System of 2 first-order ODEs:

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} z_2 \\ g - (k/m) z_1 \end{pmatrix}$$

- Initial conditions:

$z_1(0) = y(0)$ Initial position
$z_2(0) = y'(0)$ Initial velocity

$$\underbrace{\vec{z}'(t) = \begin{bmatrix} 0 & 1 \\ -(k/m) & 0 \end{bmatrix} \vec{z}(t) + \begin{pmatrix} 0 \\ g \end{pmatrix}}$$

$$\vec{z}'(t) = A\,\vec{z}(t) + \vec{b}$$

$$\vec{z}'(0) = \vec{z}_0$$

Linear system

All eigenvalues $\lambda_i$ of A should satisfy $\lambda_i h$ within stability region for the integration to be stable

# Concluding remarks

- Convert high-order ODEs to system of first order ODEs
- Graphical interpretation of ODE integration schemes
- Implementation of given ODE integration schemes
- Determination order of convergence ODE integration schemes:
  - Use Taylor series for the analysis
  - Higher order can be obtained using higher order differencing
  - Higher order can be obtained using higher order quadrature rules
- Determination stability regions ODE integration schemes:
  - Analyse the amplification of an initial error
  - Stable simulation when $\lambda_i h$ is within the stability region for all eigenvalues of A

$$\vec{z}'(t) = A\,\vec{z}(t) + \vec{b}$$