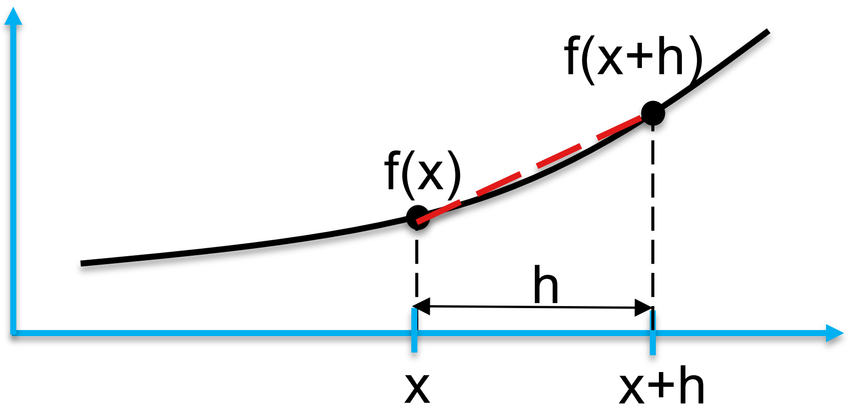
Example: approximate f'(1) using forward differencing



h	dfdx_f	error
0.1000	1.1053	0.1053
0.0100	1.0101	0.0101
0.0010	1.0010	0.0010
0.0001	1.0001	0.0001

* 0.1 * 0.1
* 0.1 * 0.1
* 0.1 * 0.1

function:

$$f(x) = x^x \quad \Longrightarrow \quad f'(1) = 1$$

forward difference:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

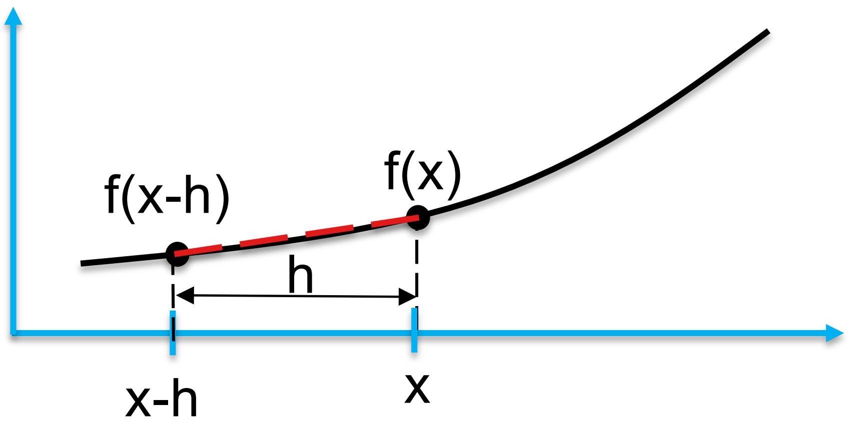
error:

$$\varepsilon(h) = \frac{f(1+h) - f(1)}{h} - f'(1)$$

Note: when step size *h* is reduced by factor 10, also the error reduces ≈ by factor 10

Error is of $O(h)$

Example: approximate f'(1) using backward differencing



function:

$$f(x) = x^x \quad \Longrightarrow \quad f'(1) = 1$$

backward difference:

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

error:

$$\varepsilon(h) = \frac{f(1) - f(1-h)}{h} - f'(1)$$

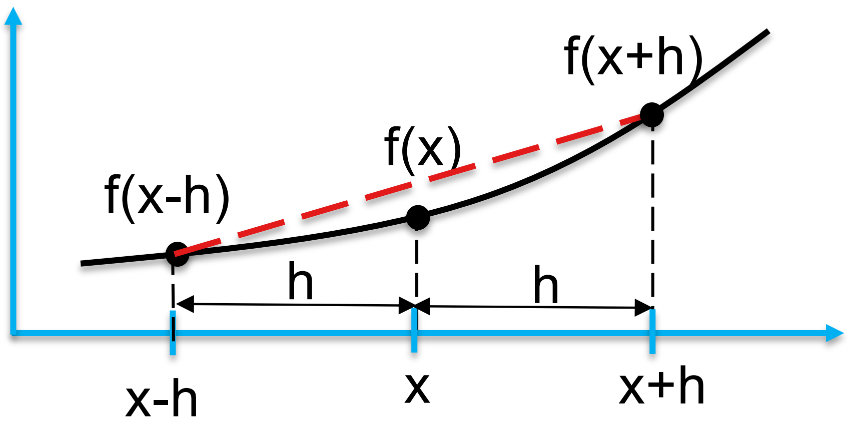
h	dfdx_b	error
0.1000	0.9047	-0.0953
0.0100	0.9900	-0.0100
0.0010	0.9990	-0.0010
0.0001	0.9999	-0.0001

* 0.1
* 0.1
* 0.1

Note: when step size *h* is reduced by factor 10, also the error reduces $\approx$ by factor 10

Error is of $O(h)$

Example: approximate f'(1) using central differencing



function:

$$f(x) = x^x \quad \Longrightarrow \quad f'(1) = 1$$

central difference:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

error:

$$\varepsilon(h) = \frac{f(1+h) - f(1-h)}{h} - f'(1)$$

h	dfdx_c	error
1.0000e-01	1.0050e+00	5.0083e-03
1.0000e-02	1.0001e+00	5.0001e-05
1.0000e-03	1.0000e+00	5.0000e-07
1.0000e-04	1.0000e+00	4.9998e-09

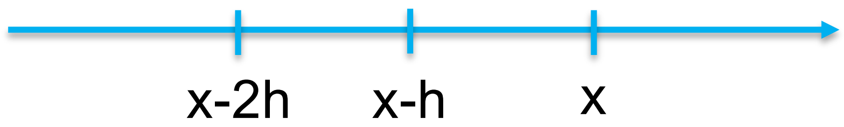
* 0.1 * 0.01
* 0.1 * 0.01
* 0.1 * 0.01

Note: when step size h is reduced by factor 10, also the error reduces $\approx$ by factor 100

Error is of $O(h^2)$

Example: constructing difference scheme for f'(x)

Choose nodes that we want to use:



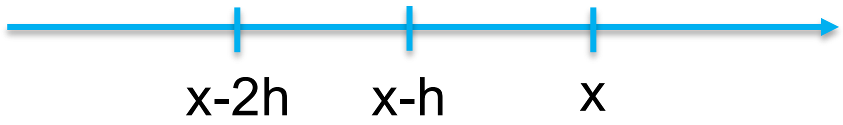
Make the Taylor expansions of *f* for each node:

$$f(x) \qquad\quad = f(x)$$

$$f(x-h) \quad = f(x) + (-1)hf'(x) + \frac{1}{2}(-1)^2h^2f''(x) + O(h^3)$$

$$f(x-2h) = f(x) + (-2)hf'(x) + \frac{1}{2}(-2)^2h^2f''(x) + O(h^3)$$

Example: constructing difference scheme for f'(x)



Make the Taylor expansions of *f* for each node:

$$f(x) \quad = f(x)$$

$$f(x-h) \quad = f(x) + (-1)hf'(x) + \frac{1}{2}(-1)^2h^2f''(x) + O(h^3)$$

$$f(x-2h) = f(x) + (-2)hf'(x) + \frac{1}{2}(-2)^2h^2f''(x) + O(h^3)$$

	$af(x-2h)$	$bf(x-h)$	$cf(x)$	
$f(x)$				
$hf'(x)$				
$h^2f''(x)$				
$O(h^3)$				

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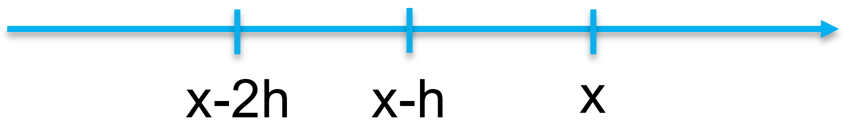
Example: constructing difference scheme for f'(x)



Make the Taylor expansions of *f* for each node:

$$f(x) \quad\quad = f(x)$$

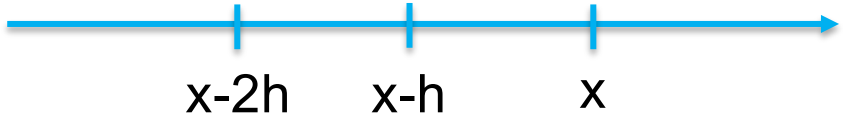
$$f(x-h) \quad = f(x) + (-1)hf'(x) + \frac{1}{2}(-1)^2h^2f''(x) + O(h^3)$$

$$f(x-2h) = f(x) + (-2)hf'(x) + \frac{1}{2}(-2)^2h^2f''(x) +$$

Fill in the "target", i.e. here we want f'(x)

	$af(x-2h)$	$bf(x-h)$	$cf(x)$	
$f(x)$				0
$hf'(x)$				1
$h^2f''(x)$				0
$O(h^3)$				

Example: constructing difference scheme for f'(x)



Make the Taylor expansions of *f* for each node:

$$f(x) = f(x)$$

$$f(x-h) = f(x) + (-1)hf'(x) + \frac{1}{2}(-1)^2h^2f''(x) + O(h^3)$$

$$f(x-2h) = f(x) + (-2)hf'(x) + \frac{1}{2}(-2)^2h^2f''(x) + O(h^3)$$

Fill in the Taylor components

	$af(x-2h)$	$bf(x-h)$	$cf(x)$	
$f(x)$	$1a$			0
$hf'(x)$	$(-2)a$			1
$h^2f''(x)$	$\frac{1}{2}(-2)^2a$			0
$O(h^3)$				

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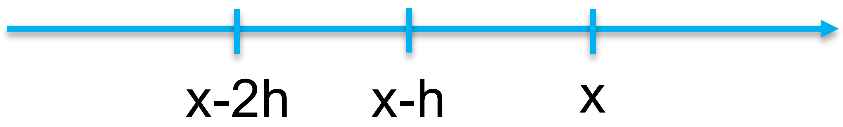
Example: constructing difference scheme for f'(x)



Make the Taylor expansions of *f* for each node:

$$f(x) \qquad = f(x)$$

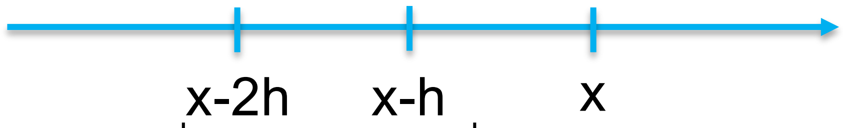
$$f(x-h) \quad = f(x) + (-1)hf'(x) + \frac{1}{2}(-1)^2h^2f''(x) + O(h^3)$$

$$f(x-2h) = f(x) + (-2)hf'(x) + \frac{1}{2}(-2)^2h^2f''(x) + O(h^3)$$

Fill in the Taylor components

	$af(x-2h)$	$bf(x-h)$	$cf(x)$	
$f(x)$	$1a$	$1b$	$1c$	0
$hf'(x)$	$(-2)a$	$(-1)b$	$0c$	1
$h^2f''(x)$	$\frac{1}{2}(-2)^2a$	$\frac{1}{2}(-1)^2b$	$0c$	0
$O(h^3)$				

Example: constructing difference scheme for f’(x)



	$af(x-2h)$	$bf(x-h)$	$cf(x)$	
$f(x)$	$1a$	$1b$	$1c$	0
$hf'(x)$	$(-2)a$	$(-1)b$	$0c$	1
$h^2f''(x)$	$\frac{1}{2}(-2)^2a$	$\frac{1}{2}(-1)^2b$	$0c$	0
$O(h^3)$				

Solve system to find a, b, c:

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$a = \frac{1}{2} \quad b = -2 \quad c = \frac{3}{2}$$

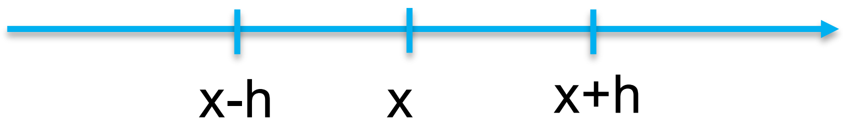
$$hf'(x) \approx \frac{1}{2}f(x-2h) - 2f(x-h) + \frac{3}{2}f(x)$$

$O(h^3)$

$$\boxed{f'(x) \approx \frac{f(x-2h) - 4f(x-h) + 3f(x)}{2h}}$$

$O(h^2)$

Example: constructing difference schemes



Make the Taylor expansions of *f* for each node:

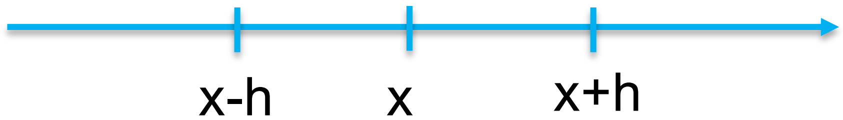
$$f(x-h) = f(x) + (-1)hf'(x) + \frac{1}{2}(-1)^2h^2f''(x) + O(h^3)$$

$$f(x) \qquad = f(x)$$

$$f(x+h) = f(x) + (+1)hf'(x) + \frac{1}{2}(+1)^2h^2f''(x) + O(h^3)$$

	$af(x-h)$	$bf(x)$	$cf(x+h)$
$f(x)$	$1a$	$1b$	$1c$
$hf'(x)$	$(-1)a$	$0b$	$(+1)c$
$h^2f''(x)$	$\frac{1}{2}(-1)^2a$	$0b$	$\frac{1}{2}(+1)^2c$

Example: constructing difference schemes – f'(x)



	$af(x-h)$	$bf(x)$	$cf(x+h)$	
$f(x)$	$1a$	$1b$	$1c$	0
$hf'(x)$	$(-1)a$	$0b$	$(+1)c$	1
$h^2f''(x)$	$\frac{1}{2}(-1)^2a$	$0b$	$\frac{1}{2}(+1)^2c$	0

If we want f'(x):

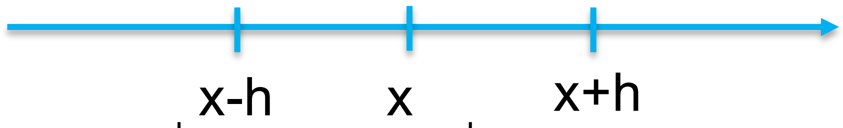
$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$a = -\frac{1}{2} \quad b = 0 \quad c = \frac{1}{2}$$

$$hf'(x) \approx -\frac{1}{2}f(x-h) + \frac{1}{2}f(x+h)$$

$$\boxed{f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}}$$

Example: constructing difference schemes – f’’(x)



	$af(x-h)$	$bf(x)$	$cf(x+h)$	
$f(x)$	$1a$	$1b$	$1c$	0
$hf'(x)$	$(-1)a$	$0b$	$(+1)c$	0
$h^2f''(x)$	$\frac{1}{2}(-1)^2a$	$0b$	$\frac{1}{2}(+1)^2c$	1

If we want f’’(x):

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$a = 1 \quad b = -2 \quad c = 1$$

$$h^2 f''(x) \approx f(x-h) - 2f(x) + f(x+h)$$

$$\boxed{f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}}$$

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